Fakultät für Physik

UNIVERSITÄT DUISBURG ESSEN

A New Universality Emerging in a Universality

Thomas Guhr

Spectral Theory and Probability in Mathematical Physics

IRMA, Université de Strasbourg, November 19th–21st, 2025

Collaborators and Publications

S. Kumar, A. Nock, H.J. Sommers, TG, B. Dietz,
M. Miski–Oglu, A. Richter, F. Schäfer,
Phys. Rev. Lett. **111** (2013) 030403

A. Nock, S. Kumar, H.-J. Sommers, TG,
Ann. Phys. **342** (2014) 103

S. Kumar, B. Dietz, TG, A. Richter,
Phys. Rev. Lett. **119** (2017) 244102

S. Köhnes, N. Gluth, B. Dietz, TG,
to be submitted (2025)

N. Gluth, A. Aldabag, S. Köhnes, B. Dietz, TG,
in preparation (2025)

supported by German Research Foundation (DFG)

Outline

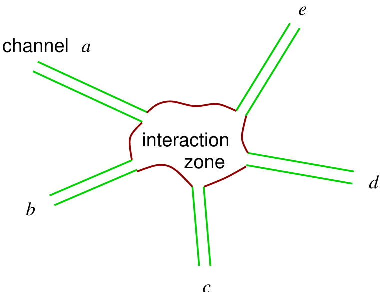
- some background: scattering theory
- (quantum) chaotic or stochastic scattering
- supersymmetry for distributions
- exact results in threefold way for off–diagonal scattering matrix elements and cross sections
- exact results for the Ericson transition
- new universality emerges in a universality
- comparison with microwave experiments and nuclear data

Introduction to Scattering Theory

Scattering Process

waves propagate in (fictitious) channels, scattered at target

scattering matrix S connects ingoing and outgoing waves



M channels,

S is $M \times M$

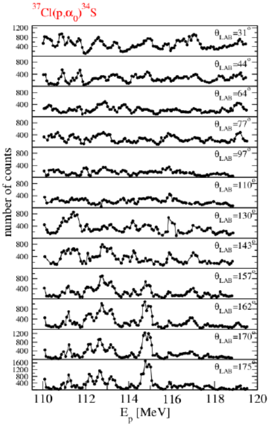
flux conservation

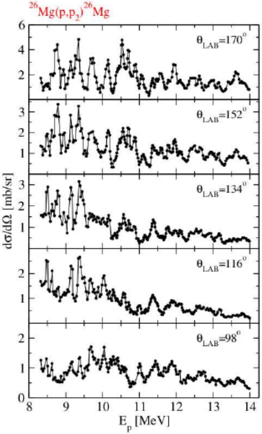
$SS^\dagger = \mathbb{1}_M = S^\dagger S$

no direct reactions ($a \neq b$) $\longrightarrow$ energy average $\overline{S}$ diagonal

transmission coefficients $T_a = 1 - |\overline{S_{aa}}|^2$

Scattering Experiments in Nuclear Physics



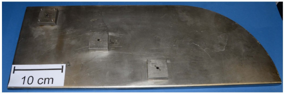


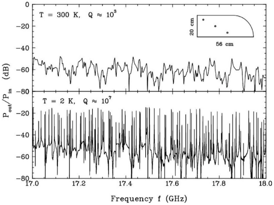
differential cross sections, squares of scattering matrix elements

from isolated resonances towards Ericson regime of strongly overlapping resonances

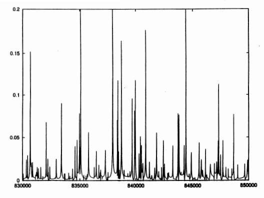
this example: Richter et al. (1960's)

Scattering Experiments with Classical Waves





microwaves



elastic reveberations

direct measurement of the scattering matrix

Weaver, Ellegaard, Stöckmann, Richter, Shridar groups (90's...10's)

(Quantum) Stochastic/Chaotic Scattering

Mexico Approach to Stochastic Scattering

to study statistics, S itself modeled as a stochastic quantity

minimum information principle yields probability measure

$$P(S)d\mu(S) \sim \frac{d\mu(S)}{|\det^{\beta(M-1)+2}(\mathbb{1}_M - S\langle S\rangle^\dagger)|}$$

- no invariance under time–reversal: S unitary, $\beta = 2$
- invariance under time–reversal $\mathcal{T}$:
  - $\mathcal{T}^2 = +\mathbb{1}$, S unitary symmetric, $\beta = 1$
  - $\mathcal{T}^2 = -\mathbb{1}$, S unitary self–dual, $\beta = 4$

input: ensemble average $\langle S\rangle$, assume $\langle S\rangle = \overline{S}$

problem: energy and parameter dependence not clear !

Microscopic Description of Scattering Process ...

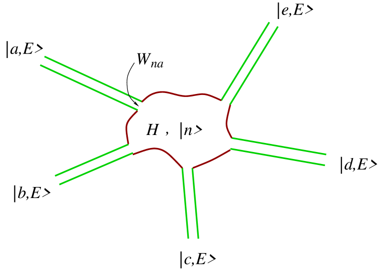
$$\mathcal{H} = \sum_{n,m=1}^{N} |n\rangle\, H_{nm}\, \langle m| + \sum_{a=1}^{M} \int dE\, |a, E\rangle\, E\, \langle a, E|$$
$$+ \sum_{n,a} \left(|n\rangle \int dE\, W_{na}\, \langle a, E| + \mathrm{c.c.} \right)$$

bound states
Hamiltonian H

$N \gg 1$ bound states $|n\rangle$

M channel states $|a, E\rangle$

coupling W_{na}



... Yields Scattering Matrix

$$S_{ab}(E) \; = \; \delta_{ab} - i2\pi W_a^\dagger G(E) W_b$$

with matrix resolvent containing bound states Hamiltonian H

$$G(E) \; = \; \frac{\mathbb{1}_N}{E\mathbb{1}_N - H + i\pi \sum_{c=1}^{M} W_c W_c^\dagger}$$

absence of direct reactions consistent with orthogonality

$$W_a^\dagger W_b \; = \; \frac{\gamma_a}{\pi} \delta_{ab}$$

Mahaux, Weidenmüller (1969)

Heidelberg Approach to Stochastic Scattering

Hamiltonian H modeled as a Gaussian random matrix

$$P(H) \sim \exp\left(-\frac{N\beta}{4v^2}\operatorname{tr} H^2\right)$$

form of $P(H)$ irrelevant on local scale of mean level spacing

$\longrightarrow$ two universalities, experimental and mathematical

- no invariance under time–reversal: H Hermitean, $\beta = 2$
- invariance under time–reversal $\mathcal{T}$:
  - $\mathcal{T}^2 = +\mathbb{1}$, H real symmetric, $\beta = 1$
  - $\mathcal{T}^2 = -\mathbb{1}$, H Hermitean self–dual, $\beta = 4$

apply Gaussian Orthogonal/Unitary/Symplectic Ensembles

Wigner (1950's), Dyson (1960's), Weidenmüller (1960's), ...

Supersymmetry for Correlations

Correlation Functions in RMT

Gaussian ensemble ($\beta = 1, 2, 4$) of $N \times N$ random matrices H

k–level correlations are probability density to find a level in each interval $[x_p, x_p + dx_p]$, $p = 1, \ldots, k$

can be expressed with resolvent

$$R_k^{(\beta)}(x_1, \ldots, x_k) \; = \; \int d[H] \exp(-\mathrm{tr}\, H^2) \prod_{p=1}^{k} \mathrm{tr} \, \frac{\mathbb{1}_N}{H - x_p \mathbb{1}_N}$$

(notation is a bit simplified)

Generating Function for Correlations

introduce scalar source variables J_p

$$R_k^{(\beta)}(x_1,\ldots,x_k) \; = \; \frac{\partial^k}{\prod_{p=1}^k \partial J_p} Z_k^{(\beta)}(x+J)\bigg|_{J=0}$$

and generating function

$$Z_k^{(\beta)}(x+J) \; = \; \int d[H] \exp(-\mathrm{tr}\, H^2) \prod_{p=1}^{k} \frac{\det(H-x_p-J_p)}{\det(H-x_p+J_p)}$$

Supersymmetric Representation

vectors z_p, ζ_p with commuting and anticommuting entries

$$\frac{\det(H - x_p - J_p)}{\det(H - x_p + J_p)} = \int d[z_p] \exp\left(iz_p^\dagger(H - x_p + J_p)z_p\right) \int d[\zeta_p] \exp\left(i\zeta_p^\dagger(H - x_p - J_p)\zeta_p\right)$$

average over H just Gaussian!

intermediate result is integral over $z_p, \zeta_p,\ p = 1, \ldots, k$

but depends only on scalar products

$z_p^\dagger z_q,\ \zeta_p^\dagger \zeta_q$ commuting

$z_p^\dagger \zeta_q,\ \zeta_p^\dagger z_q$ anticommuting

Supermatrix Integral

Hubbard–Stratonovitch transformation: “use scalar products as integration variables, remaining ones are trivial”

identity (yes, this is exact) for generating function

$$Z_k^{(\beta)}(x+J) \; = \; \int d[\sigma]\exp(-\mathrm{str}\,\sigma^2)\mathrm{sdet}^{\,-N}(\sigma - x - J)$$

where σ is a $2k \times 2k$ or $4k \times 4k$ supermatrix

$\longrightarrow$ drastic reduction of dimensions

Efetov (1983), Verbaarschot, Zirnbauer (1985), TG (1991,2006)

Scattering Matrix

$$S_{ab}(E) \; = \; \delta_{ab} \; - \; i2\pi W_a^\dagger G(E) W_b$$

does not depend on an invariant, but on resolvent matrix

$$G(E) \; = \; \frac{\mathbb{1}_N}{E\mathbb{1}_N - H + i\pi \sum_{c=1}^{M} W_c W_c^\dagger}$$

introduce $N \times N$ matrix source variable J

$$G_{nm}(E) \; = \; \frac{\partial}{\partial J_{nm}} \frac{\det(G^{-1}(E) - J)}{\det(G^{-1}(E) + J)} \Bigg|_{J=0}$$

determinants linear in H $\longrightarrow$ supersymmetry method

Verbaarschot, Weidenmüller, Zirnbauer (1985)

Many Results Obtained in this Way, for Example

two–point correlation functions $\langle S^*_{ab}(E_1) S_{cd}(E_2) \rangle$

$\beta = 1$ Verbaarschot, Weidenmüller, Zirnbauer (1985)
$\beta = 2$ Savin, Fyodorov, Sommers (2006)

higher order correlations, perturbative time–invariance breaking

Davis, Boosé (1988, 1989), Davis, Hartmann (1990)

distribution of diagonal elements $P(S_{aa}(E))$

Fyodorov, Savin, Sommers (2005)

... but: does not work for distribution $P(S_{ab}(E)), \ a \neq b$

$\longrightarrow$ **new method needed**

New Variant of the Supersymmetry Method:

Supersymmetry for Distributions

Distribution of Scattering Matrix Elements

$$S_{ab}(E) \;=\; \delta_{ab} \,-\, i2\pi W_a^\dagger G(E) W_b$$

wish to calculate distribution of real and imaginary part

$$\wp_s(S_{ab}) \;=\; \pi\big((-i)^s W_a^\dagger G W_b + i^s W_b^\dagger G^\dagger W_a\big)$$

such that

$$x_1 = \wp_1(S_{ab}) = \mathrm{Re}\, S_{ab}(E) \quad \text{and} \quad x_2 = \wp_2(S_{ab}) = \mathrm{Im}\, S_{ab}(E)$$

distribution given by

$$P_s(x_s) \;=\; \int d[H] \exp(-\mathrm{tr}\, H^2) \delta(x_s - \wp_s(S_{ab})) \;, \quad s = 1, 2$$

Characteristic Function

obtain distribution by Fourier backtransform of

$$R_s(k) \;=\; \int d[H] \exp(-\mathrm{tr}\, H^2) \exp(-ik\wp_s(S_{ab}))$$

insert definition of scattering matrix

$$R_s(k) \;=\; \int d[H] \exp(-\mathrm{tr}\, H^2) \exp(-ik\pi W^\dagger A_s W)$$

with $\quad W \;=\; \begin{bmatrix} W_a \\ W_b \end{bmatrix} \quad$ and $\quad A_s \;=\; \begin{bmatrix} 0 & (-i)^s G \\ i^s G^\dagger & 0 \end{bmatrix}$

where A_s Hermitean, but contains H inverse

problem: have to invert A_s to perform H average !

Crucial Trick

Fourier transform in W space ! — Yields

$$\exp(-ik\pi W^\dagger A_s W) \sim \int d[z] \exp\left(\frac{i}{2}(W^\dagger z + z^\dagger W)\right) \det{}^{\beta/2} A_s^{-1} \exp\left(\frac{i}{4\pi k} z^\dagger A_s^{-1} z\right)$$

now use anticommuting variables

$$\det{}^{\beta/2} A_s^{-1} \sim \int d[\zeta] \exp\left(\frac{i}{4\pi k} \zeta^\dagger A_s^{-1} \zeta\right)$$

now H linear in exponent $\longrightarrow$ **supersymmetry applicable !**

different role of commuting and anticommuting variables

Supermatrix Model

Hubbard–Stratonovitch transformation gives

$$R_s(k) \;=\; \int d[\sigma]\exp\Big(-r\,\mathrm{str}\,\sigma^2 - \frac{\beta}{2}\mathrm{str}\,\ln\boldsymbol{\Sigma} - \frac{i}{4}F_s\Big)$$

with $2k \times 2k$ or $4k \times 4k$ supermatrix σ and $r = 4\beta\pi^2 k^2 N/v^2$

$$\boldsymbol{\Sigma} \;=\; \sigma_E \otimes \mathbb{1}_N + \frac{i}{4k}L \otimes \sum_{c=1}^{M} W_c W_c^\dagger \;, \quad \sigma_E = \sigma - \frac{E}{4\pi k}\mathbb{1}_{8/\beta}$$

matrix L is some superspace metrik

F_s apart from details $W^\dagger\boldsymbol{\Sigma}^{-1}W$, projects onto boson–boson space

$\longrightarrow$ symmetry breaking differs from the one for correlations

Supersymmetric Non–Linear sigma Model

limit $N \longrightarrow \infty$, unfolding by saddlepoint approximation
integrate out "massive" modes

left with integral over "Goldstone" modes Q,
free rotations, coset manifold in superspace

$$R_s(k) \ = \ \int d\mu(Q) \exp\left(-\frac{i}{4} F_s\right) \prod_{c=1}^{M} \text{sdet}^{-\beta/2}\left(\mathbb{1}_{8/\beta} + \frac{i\gamma_c}{4\pi k} Q_E^{-1} L\right)$$

integrate out all remaining anticommuting variables

left with ordinary integrals, two for $\beta = 2$, four for $\beta = 1$

$\longrightarrow$ drastically reduced number of integration variables

Similarity and Difference to Case of Correlations

structure of non–linear sigma model very similar to the one in Verbaarschot, Weidenmüller, Zirnbauer (1985) for correlation functions $\langle S^*_{ab}(E_1) S_{cd}(E_2) \rangle$

supergroup structure and hyperbolic symmetry (noncompactness) are the same, coset manifolds

$\beta = 2 \qquad \mathrm{U}(1,1|2)/(\mathrm{U}(1|1) \times \mathrm{U}(1|1))$

$\beta = 1 \qquad \mathrm{UOSp}(2,2|4)/(\mathrm{UOSp}(2|2) \times \mathrm{UOSp}(2|2))$

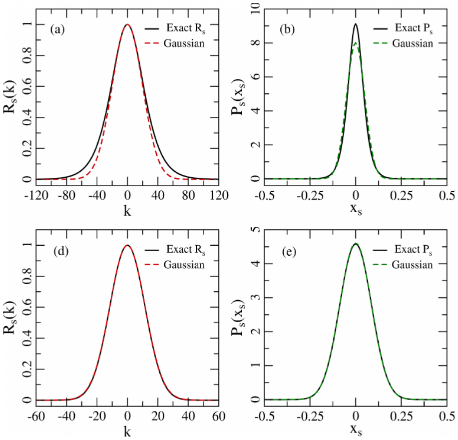
but for different reasons

supersymmetry breaking not the same: imbalance, different roles for commuting and anticommuting variables

$F_s \ \sim \ [W^\dagger \ 0^\dagger] \, \mathbf{\Sigma}^{-1} \begin{bmatrix} W \\ 0 \end{bmatrix}$, projects onto boson–boson space only

Analytical Results versus Numerics

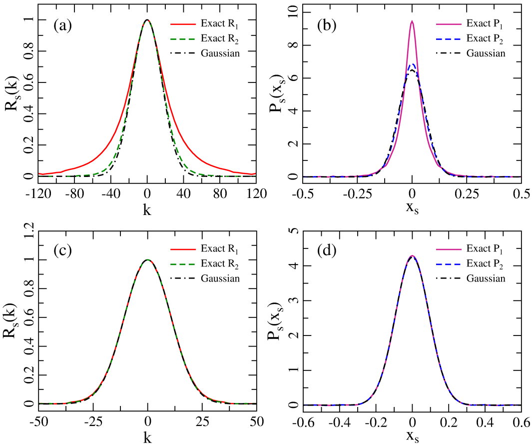
Towards Ericsson Regime for $\beta = 2$



average resonance width / mean level spacing $\Gamma/D = 0.716$ (top) and $\Gamma/D = 8.594$ (bottom)

real and imaginary parts equally distributed for $\beta = 2$

Towards Ericsson Regime for $\beta = 1$

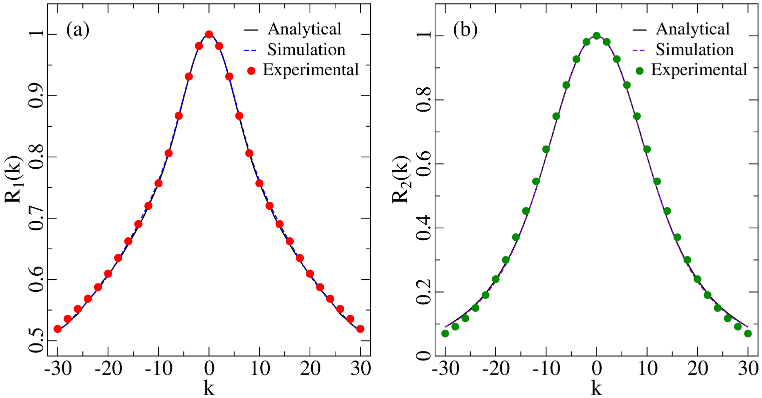


average resonance width / mean level spacing $\Gamma/D = 1.273$ (top) and $\Gamma/D = 7.162$ (bottom)

real and imaginary parts not equally distributed for $\beta = 1$

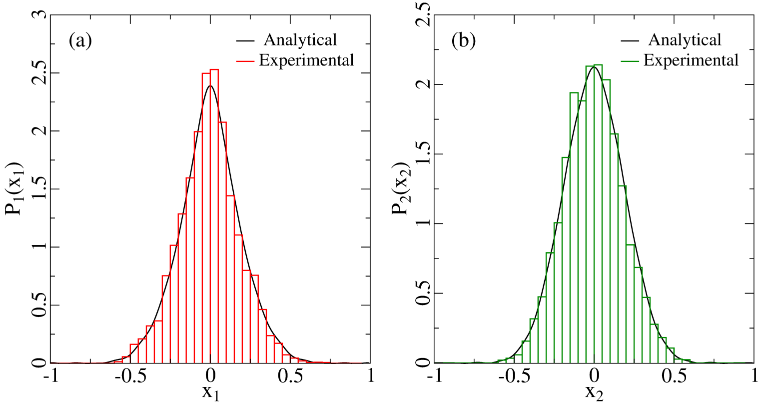
Comparison with Microwave Experiments

... vs Numerics and Experiment for $\beta = 1$



frequency range $10 \ldots 11\mathrm{GHz}$,
average resonance width / mean level spacing $\Gamma/D = 0.234$

Analytical Result vs Experiment for $\beta = 1$



frequency range $24\ldots25\mathrm{GHz}$,
average resonance width / mean level spacing $\Gamma/D = 1.21$

Distribution of Cross Sections

Joint Probability Density Needed

cross section $\sigma_{ab}(E) = |S_{ab}(E)|^2 = \mathrm{Re}^2 S_{ab}(E) + \mathrm{Im}^2 S_{ab}(E)$

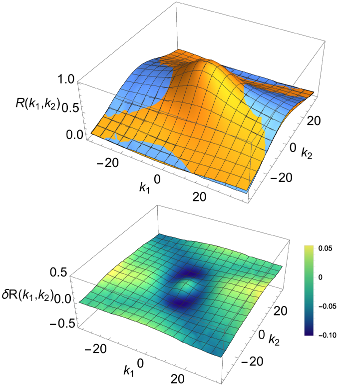
joint pdf $P(\mathrm{Re}\, S_{ab}, \mathrm{Im}\, S_{ab}) = P(S_{ab}, S_{ab}^*)$

good news: can extend previous calculation into complex plane

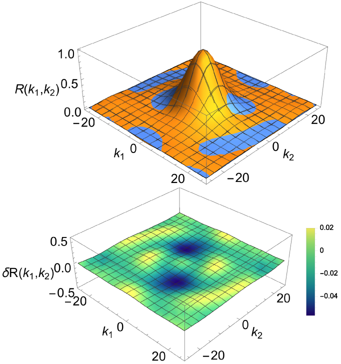
bivariate characteristic function $R(k, k^*)$

distribution $p(\sigma_{ab}(E)) = \int d^2k\, R(k, k^*)\, J_0(\sqrt{\sigma_{ab}(E)}|k|)$

Characteristic Functions for Microwave Data

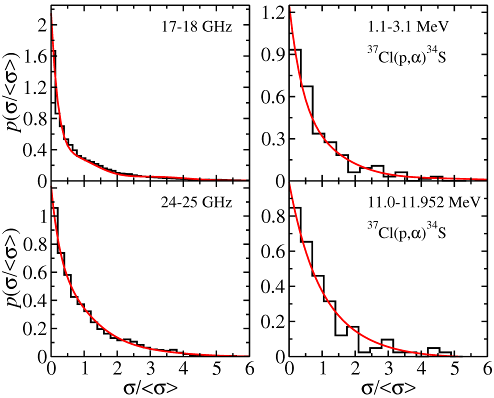


$$\Gamma/D = 0.234$$



$$\Gamma/D = 1.21$$

Analytics vs Microwave, Nuclear Cross Section Data

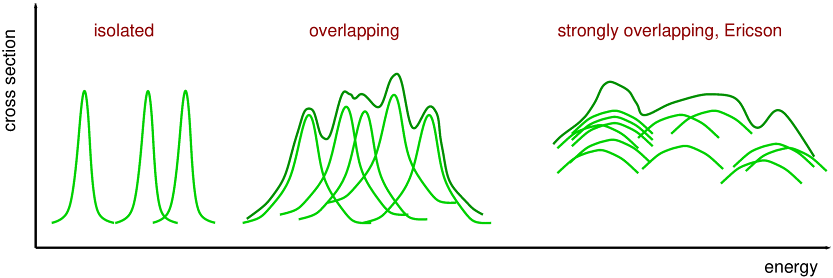


left: microwave data, $\Gamma/D = 0.7,\ 1.2$
right: nuclear data, $\Gamma/D \approx 1,\ 30$

$p(0) \approx 1$ indicates Ericson regime

Transition to the Ericson Regime

Cartoon of the Ericson Transition



average resonance width / average mean level spacing $= \Gamma/D$

Pedestrian–Type–of Motivation

scattering matrix element as function of energy E something like

$$S_{ab}(E) \sim \sum_r \frac{A_r}{E - E_r - i\Gamma_r/2}$$

cross section is absolute value squared

$$\begin{aligned}\sigma_{ab}(E) &= |S_{ab}(E)|^2 \\ &\sim \sum_r \frac{|A_r|^2}{(E-E_r)^2 + \Gamma_r^2/4} \\ &\qquad + 2\mathrm{Re} \sum_{r<r'} \frac{A_r A_{r'}}{(E-E_r+i\Gamma_r/2)(E-E_{r'}-i\Gamma_{r'}/2)}\end{aligned}$$

Γ/D small $\longrightarrow$ only first term relevant $\longrightarrow$ isolated resonances

when Γ/D gets larger $\longrightarrow$ second term also becomes relevant $\longrightarrow$ resonances start overlapping

Distributions in the Ericson Regime

Ericson (early 1960's): if Γ/D very large, many open channels, and transmission coefficients T_c comparable, then

real and imaginary parts of $S_{ab}(E)$ are Gaussian distributed

cross sections $\sigma_{ab}(E)$ are exponentially distributed

heuristic reasoning based on Central–Limit–type–of arguments, neither analytical derivation nor analytical understanding of the transition — when does it set in?

finally completely and exactly solved

new universality emerges out of the universal setting of quantum chaotic/stochastic scattering

Köhnes, TG (2025)

Sketch of Exact Derivation

Weisskopf Estimate and Transmission Coefficients

Weisskopf estimate relates transmission coefficients to parameter

$$\Xi \;=\; \frac{\Gamma}{D} \;=\; \frac{1}{2\pi}\sum_{c=1}^{M} T_c$$

consider infinite number M of channels where $T_c \;\sim\; \frac{1}{M}$

$\longrightarrow\;\; \Xi \;=\; \frac{\Gamma}{D}$ very large

$\longrightarrow$ asymptotics by expanding in $\frac{1}{\Xi}$

Characteristic Function ...

unitary case, characteristic function double integral

$$R_s(k) \ = \ 1 - \int_1^\infty d\lambda_1 \int_{-1}^{1} d\lambda_2 \frac{k^2 \mathcal{F}_U(\lambda_1,\lambda_2)}{4(\lambda_1-\lambda_2)^2} \left(t_a^1 t_b^1 + t_a^2 t_b^2\right) J_0(k\sqrt{t_a^1 t_b^1})$$

$s = 1, 2$ for real and imaginary part, equally distributed

$$g_c^+ = 2/T_c - 1 \quad , \qquad t_c^j = \sqrt{|\lambda_j^2 - 1|}/(g_c^+ + \lambda_j)$$

channel factor $\qquad \mathcal{F}_U(\lambda_1,\lambda_2) = \prod_{c=1}^{M} \frac{g_c^+ + \lambda_2}{g_c^+ + \lambda_1}.$

... Generates All Moments

$$R_s(k) \;=\; \sum_{n=0}^{\infty}(-1)^n \frac{k^{2n}}{(2n)!}\overline{x_s^{2n}}$$

with $\quad x_1 = \mathrm{Re}\, S_{ab} \quad , \quad x_2 = \mathrm{Im}\, S_{ab}$

all moments exist, because scattering matrix is unitary and thus its elements bounded above

Moments and their Asymptotics

channel factor in the asymptotic limit

$$\mathcal{F}_U(\lambda_1, \lambda_2) \;\longrightarrow\; \exp\left(-\pi\Xi(\lambda_1 - \lambda_2)\right)$$

parameter $\Xi = \Gamma/D$ pops up !

first and second leading terms of moments

$$\overline{x_s^{2n}} \;=\; \frac{(2n)!}{n!}\left(\frac{1}{2(g_a^+ + 1)(g_b^+ + 1)\pi\Xi}\right)^n + \frac{g_a^+ g_b^+ - g_a^+ - g_b^+ - 3}{(2(g_a^+ + 1)(g_b^+ + 1)\pi\Xi)^{n+1}}\frac{\Gamma(2n+1)}{\Gamma(n-1)} \;+\; \mathcal{O}\left(\frac{1}{\Xi^{n+2}}\right)$$

with $x_1 = \mathrm{Re}\, S_{ab}, \;\; x_2 = \mathrm{Im}\, S_{ab}$

Distribution of Real and Imaginary Parts of $S_{ab}(E)$

resummation of characteristic function, then Fourier backtransform

distribution is Gaussian with leading correction

$$P_s(x_s) \simeq \sqrt{\frac{\Xi(g_a^+ + 1)(g_b^+ + 1)}{2}} \exp\left(-\frac{\pi(g_a^+ + 1)(g_b^+ + 1)\Xi x_s^2}{2}\right)$$
$$\left(1 + \left(3 - 6\pi(g_a^+ + 1)(g_b^+ + 1)\Xi x_s^2\right.\right.$$
$$\left.\left. + (\pi(g_a^+ + 1)(g_b^+ + 1))^2 \Xi^2 x_s^4\right) \frac{g_a^+ g_b^+ - g_a^+ - g_b^+ - 3}{8(g_a^+ + 1)(g_b^+ + 1)\pi\Xi}\right)$$

Distribution Properly Rescaled

rescaling $\xi_s = \sqrt{\Xi} x_s$

$$P_s(\xi_s) \simeq \sqrt{\frac{(g_a^+ + 1)(g_b^+ + 1)}{2}} \exp\left(-\frac{\pi (g_a^+ + 1)(g_b^+ + 1)\xi_s^2}{2}\right)$$
$$\left(1 + \left(3 - 6\pi (g_a^+ + 1)(g_b^+ + 1)\xi_s^2\right.\right.$$
$$\left.\left. + (\pi (g_a^+ + 1)(g_b^+ + 1))^2 \xi_s^4\right) \frac{g_a^+ g_b^+ - g_a^+ - g_b^+ - 3}{8 (g_a^+ + 1)(g_b^+ + 1)\pi \Xi}\right)$$

pure $\frac{1}{\Xi} = \frac{1}{\Gamma / D}$ **asymptotics**

Distribution of Cross Sections

rescaling $\widetilde{\sigma}_{ab}(E) = \Xi\sigma_{ab}(E)$

distribution is exponential with leading correction

$$p(\widetilde{\sigma}_{ab}) \;\simeq\; \frac{1}{2}\exp\left(-\frac{(g_a^+ + 1)(g_b^+ + 1)\pi\widetilde{\sigma}_{ab}}{2}\right)$$

$$\left((g_a^+ + 1)(g_b^+ + 1)\pi + (g_a^+ g_b^+ - g_a^+ - g_b^+ - 3)\right.$$

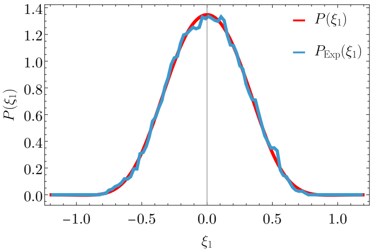
$$\left.\left(1 - (g_a^+ + 1)(g_b^+ + 1)\pi\widetilde{\sigma}_{ab} + \frac{1}{8}((g_a^+ + 1)(g_b^+ + 1)\pi\widetilde{\sigma}_{ab})^2\right)\frac{1}{\Xi}\right)$$

value at zero

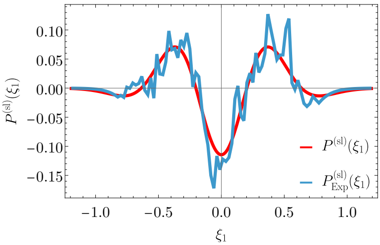
$$p(0) \;\simeq\; \frac{(g_a^+ + 1)(g_b^+ + 1)\pi}{2} + \frac{g_a^+ g_b^+ - g_a^+ - g_b^+ - 3}{2\Xi}$$

Analytical Results
versus Microwave Experiments

Distribution of Scattering Matrix Elements



full distribution

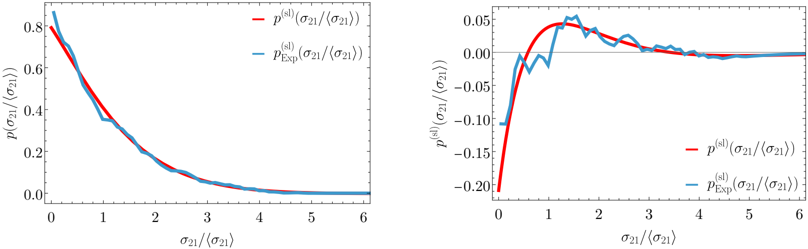


first order correction
to Gaussian

$\Xi = \Gamma/D = 1.4$, **onset of Ericson regime, correction works**

Köhnes, Dietz,TG (2025)

Distribution of Cross Sections



full distribution
$p(0) < 1$

first order correction
to exponential

$\Xi = \Gamma/D = 1.4$, **onset of Ericson regime, correction works**

Köhnes, Dietz,TG (2025)

What have we learned ?

Conclusions and Outlook

- solved longstanding problem within Heidelberg approach
- supersymmetry for distributions of off–diagonal scattering matrix elements and cross sections
- exact results in threefold way: othogonal, unitary, symplectic
- full analytical understanding of transition to Ericson regime
- transition is fast, quantitatively captured
- new universality emerging in a universality
- comparison with microwave experiments and nuclear data
- Brouwer's equivalence proof Heidelberg–Mexico implies: now have explicit handle on Mexico approach for arbitrary channel number
- also: condensed matter and wireless communication

Thank You for Your Attention !

Supermathematics and Supersymmetry

Two Kinds of Variables

k_1 complex commuting variables $\quad z_p,\ p = 1, \ldots, k_1$

k_2 complex anticommuting variables $\quad \zeta_p,\ p = 1, \ldots, k_2$

$\zeta_p \, \zeta_q \ = \ -\zeta_q \, \zeta_p$, $\qquad$ in particular $\quad \zeta_p^2 \ = \ 0$

every function is a finite polynomial, for example for $k_2 = 2$

$$f(\zeta_1, \zeta_2) \ = \ c_0 + c_{11}\zeta_1 + c_{12}\zeta_2 + c_2\zeta_1\zeta_2$$

complex conjugation $\quad \zeta_p \ \longrightarrow \ \zeta_p^* \ \longrightarrow \ \zeta_p^{**} = -\zeta_p$

$$\zeta_p \, \zeta_q^* \ = \ -\zeta_q^* \, \zeta_p$$

commuting and anticommuting variables commute

$$z_p \, \zeta_q \ = \ \zeta_q \, z_p \quad \text{and} \quad z_p \, \zeta_q^* \ = \ \zeta_q^* \, z_p$$

Linear Algebra in Superspace

supervectors $\psi = \begin{bmatrix} z \\ \zeta \end{bmatrix}$ and supermatrices $\sigma = \begin{bmatrix} a & \mu \\ \nu & b \end{bmatrix}$

matrices a, b have commuting entries
matrices μ, ν have anticommuting entries

$$\sigma\psi = \begin{bmatrix} a & \mu \\ \nu & b \end{bmatrix} \begin{bmatrix} z \\ \zeta \end{bmatrix} = \begin{bmatrix} az + \mu\zeta \\ \nu z + b\zeta \end{bmatrix} = \begin{bmatrix} z' \\ \zeta' \end{bmatrix} = \psi'$$

supertrace $\quad \operatorname{str}\sigma = \operatorname{tr} a - \operatorname{tr} b \quad \longrightarrow \quad \operatorname{str}\sigma_1\sigma_2 = \operatorname{str}\sigma_2\sigma_1$

superdeterminant $\quad \operatorname{sdet}\sigma = \dfrac{\det(a - \mu b^{-1}\nu)}{\det b}$

$$\longrightarrow \quad \operatorname{sdet}\sigma_1\sigma_2 = \operatorname{sdet}\sigma_1 \operatorname{sdet}\sigma_2$$

Analysis in Superspace

derivative $\quad \frac{\partial \zeta_p}{\partial \zeta_q} = \delta_{pq} \quad$ and $\quad \frac{\partial \zeta_p^*}{\partial \zeta_q} = 0$

Berezin integral $\quad \int d\zeta_p = 0 \quad$ and $\quad \int \zeta_p d\zeta_p = \frac{1}{\sqrt{2\pi}}$

for example

$$\int \exp(-a\zeta_p^*\zeta_p) d\zeta_p^* d\zeta_p = \int \left(1 - a\zeta_p^*\zeta_p\right) d\zeta_p^* d\zeta_p = \frac{a}{2\pi}$$

apart from factors, derivative and integral are the same !

change of variables $\quad \psi \ \rightarrow \ \chi = \chi(\psi) \quad$ requires

Jacobian or Berezinian $\quad \int f(\psi) d[\psi] = \int f(\psi(\chi)) \mathrm{sdet} \frac{\partial \psi}{\partial \chi} d[\chi]$

Gaussian Integrals over Supervectors

matrix a has commuting entries

$$\int \exp(-z^\dagger a z) d[z] = \det{}^{-1} \frac{a}{2\pi} \quad \text{and}$$
$$\int \exp(-\zeta^\dagger a \zeta) d[\zeta] = \det \frac{a}{2\pi}$$

σ is a supermatrix

$$\int \exp(-\psi^\dagger \sigma \psi) d[\psi] = \text{sdet}^{-1} \frac{\sigma}{2\pi}$$

$\longrightarrow$ divergencies removed $\longrightarrow$ renormalization

$\longrightarrow$ Random Matrix Theory and disordered systems

Symplectic Symmetry

New Theoretical Challenge

two time–reversal invariant classes

- with (total) spin–rotation symmetry:
  S unitary symmetric, H real symmetric, $\beta = 1$
- no spin–rotation symmetry:
  S unitary self–dual, H Hermitean self–dual, $\beta = 4$

$\beta = 4$ for mathematical reasons called **symplectic case**

the physical system must involve **spin degrees of freedom**

Kramers degeneracy of eigenvalues, typically doubly degenerate

Hamiltonian H from **Gaussian Symplectic Ensemble (GSE)**

New Experiments, New Realizations

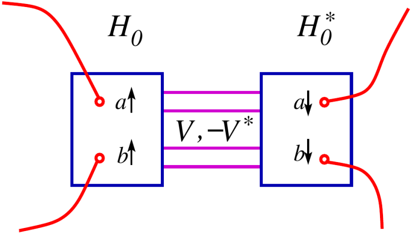
long time thought to be a mathematical curiosity of only minor physics relevance, **this changes rapidly !**

- Kuemmeth, Bolotin, Shi, Ralph (2008)
  Gold nanoparticles, level statistics is GSE due to strong spin–orbit coupling
- Joyner, Müller, Sieber (2014)
  graph has GSE statistics, no spin, rather equivalent geometry
- microwave experiments (2014–2023)
  - Kuhl/Stöckmann group (Nice/Marburg)
  - Dietz group (Lanzhou, Daejeon)
  - Sirko group (Warsaw)

Where is the Spin in the GSE ?

GSE generated by GUE matrices H_0 and π rotated H_0^* coupled by matrices V, $-V^*$

$$\begin{bmatrix} H_0 & V \\ -V^* & H_0^* \end{bmatrix}$$



also: every channel gets two spin directions $a\uparrow$, $a\downarrow$

every (!) matrix element is a 2×2 **quaternion** (Pauli matrices)

one scattering matrix element $S_{ab}(E) = \begin{bmatrix} S_{a\uparrow b\uparrow}(E) & S_{a\uparrow b\downarrow}(E) \\ S_{a\downarrow b\uparrow}(E) & S_{a\downarrow b\downarrow}(E) \end{bmatrix}$

Exact Results and Data Comparisons

Supersymmetric Non–Linear sigma Model

calculation more involved than for $\beta = 1, 2$, result similar

close structural similarity to $\beta = 1$, 8×8 supermatrices,
integral over "Goldstone" modes Q, coset manifold in superspace,
two versions of $\mathrm{UOSp}(2,2|4)/\mathrm{UOSp}(2|2) \otimes \mathrm{UOSp}(2|2)$

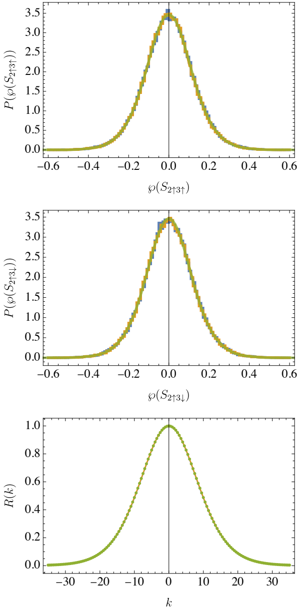
$$R_s(k) = \int d\mu(Q) \exp\left(-\frac{i}{4}F_s\right) \prod_{c=1}^{M} \mathrm{sdet}^{-2}\left(\mathbb{1}_8 + \frac{i\gamma_c}{4\pi k} Q_E^{-1} L\right)$$

integrate out all remaining anticommuting variables,
left with three ordinary integrals for $\beta = 4$

$\longrightarrow$ drastically reduced number of integration variables

Gluth, TG (2025)

Exact Results vs RMT Monte Carlo Simulation



$M = 5$, $\Gamma/D = 0.886$

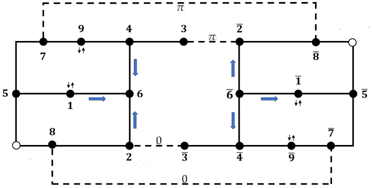
distributions of
Re $S_{2\uparrow 3\uparrow}$, Im $S_{2\uparrow 3\uparrow}$
Re $S_{2\uparrow 3\downarrow}$, Im $S_{2\uparrow 3\downarrow}$

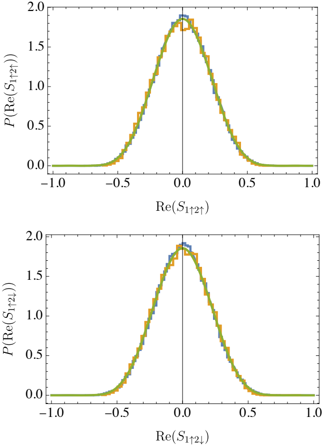
characteristic function

real and imaginary part of $S_{am\, bm'}$ equally distributed, as for $\beta = 2$, but different from $\beta = 1$

no interaction prefering spin direction in the model yet, all four components of S_{ab} equally distributed

Exact Results vs Quantum Graph Calculation





two subgraphs model H_0 and H_0^*, four couplings model V, $-V^*$

Dietz (2025)