Field Theory of Anderson (De-)Localization: Singular Continuous Spectrum

M. Zirnbauer (Uni Köln)

STROM 2025 @ IRMA, Université de Strasbourg

November 19-21, 2025

based on arXiv:2305.00243, joint work with J. Arenz

Introduction

Models

Random Schrödinger operators $H = \sum V_n c_n^+ c_n + \sum t_{n,n'} c_n^+ c_{n'} + h.c.$

- Anderson model (Anderson, 1958):
  discrete Laplacian + random on-site potential, on $\mathbb{Z}^d$.

- Wegner N-orbital model: $\mathcal{H} = \mathbb{C}^N \otimes \ell^2(\mathbb{Z}^d)$
  (original purpose: $1/N$ expansion; distinguishing feature: random hopping $\leadsto$ quantum percolation characteristics, invariance under local $U(N)$ rotations).

- Random network models
  e.g. Chalker–Coddington model (IQHT)

$$\psi_{t+1} = \mathcal{U}\psi_t\,, \quad \mathcal{U} = \mathcal{U}_{reg}\,\mathcal{U}_{ran}$$

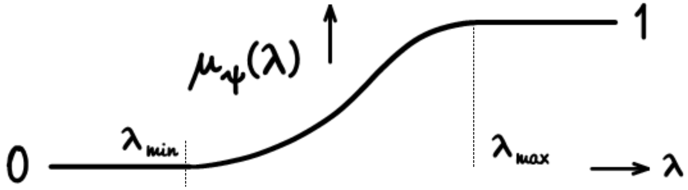
diagonal in k-space → $\mathcal{U}_{reg}$; $\mathcal{U}_{ran}$ ← diagonal in x-space

Types of Spectrum

<u>Fact.</u> For any self-adjoint operator H acting on a Hilbert space $\mathcal{H}$, one has an orthogonal decomposition: $\mathcal{H} = \mathcal{H}_{ac} \oplus \mathcal{H}_{sc} \oplus \mathcal{H}_{pp}$ [ac = absolutely continuous, sc = singular continuous, pp = pure point].

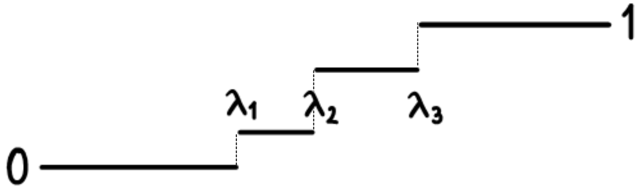
Let $H = \int_{\mathbb{R}} \lambda \, dE_H(\lambda)$, $d\mu_\psi(\lambda) = d\langle \psi, E_H(\lambda)\psi \rangle$ (Stieltjes). Then

- $\psi \in \mathcal{H}_{ac}$:



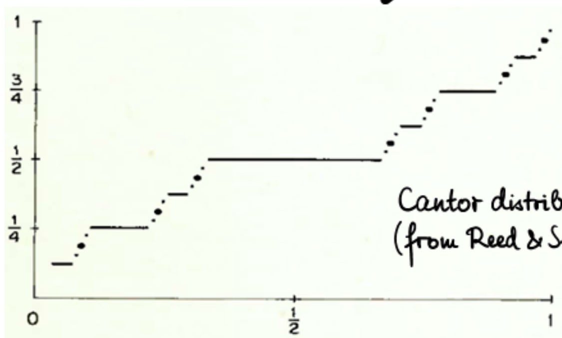
$$d\mu_\psi(\lambda) = \nu_\psi(\lambda)\, d\lambda$$

- $\psi \in \mathcal{H}_{pp}$:



$$d\mu_\psi(\lambda) = \sum_i |\langle \psi, \varphi_i \rangle|^2 \delta_{\lambda_i}$$

- $\psi \in \mathcal{H}_{sc}$: devil's stair case $\longrightarrow$



Cantor distribution (from Reed & Simon, vol. I)

<u>Note</u>:

spectrum	pp	ac	sc
eigenfcts	localized	extended	fractal

Some MP results

- **Pure point spectrum**

  Proofs of localization for strong disorder and/or low energy:

  — Fröhlich & Spencer (1983): multiscale analysis

  — Aizenman & Molchanov (1993): fractional moments method

- **Absolutely continuous spectrum**

  — Proofs for the Bethe lattice (Cayley tree):

  Kunz & Souillard (1983), A. Klein (1994), Aizenman & Warzel (2005–2015)

  — Expected in $d \geq 3$, but no rigorous result so far; see, however,

  Disertori, Spencer & MZ (2010): $H^{2|2}$ model ($\to$ classical random walk with memory)

- **Singular continuous spectrum**

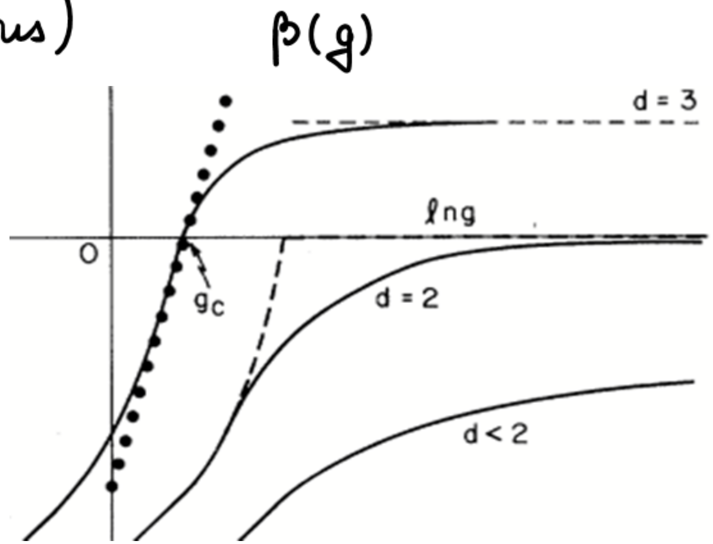
  — B. Simon (1994–1998, ~10 papers $\to$ "sc spectrum is generic")

Standard physics picture

- Abrahams, Anderson, Licciardello, Ramakrishnan:
  "Scaling theory of localization: absence of quantum diffusion in two dimensions" (PRL, 1979; $\geqslant$ 5000 citations)

$\longrightarrow$ one-parameter scaling $\longleftarrow$ $dg/d\ln L = \beta(g)$,

$$\beta(g) = g\,(d-2-a/g+\ldots).$$



- Wegner, Efetov, Hikami, ... (1979 – 1983)
  nonlinear σ-model (supermatrix Q, $Q^2=1$)

$$S[Q] = \int d^dx\ \mathrm{STr}\left(\sigma_{xx}\,\partial_\mu Q\,\partial^\mu Q + \varepsilon\nu\Sigma_3 Q\right)$$

conductivity $\sigma_{xx}\propto D\nu$, conductance $g=\sigma_{xx}L^{d-2}$.

Remark. The approximations leading to NLσM are well justified in the metallic regime. There is <u>no</u> proof of renormalizability for strong coupling.

Doubts...

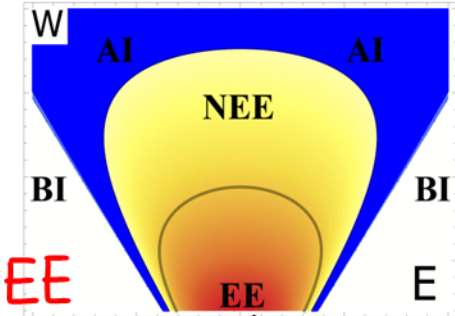
- MZ: "The integer quantum Hall plateau transition is a current algebra after all" (Nucl. Phys. B, 2019)

→ Conflict between nonlinear σ-model and conformal invariance:

$$\partial_\mu J^\mu = 0 \xRightarrow{\text{conf. inv.}} \partial_\mu \varepsilon^\mu{}_\nu J^\nu = 0 \;\overset{\text{NL}\sigma\text{M}}{\not\Longleftarrow}\; \partial_\mu J^\mu = 0.$$

- Kravtsov, Altshuler, Ioffe (Ann. Phys., 2018) →

[History: B. Shapiro; N. Goldenfeld; I.M. Suslov...]

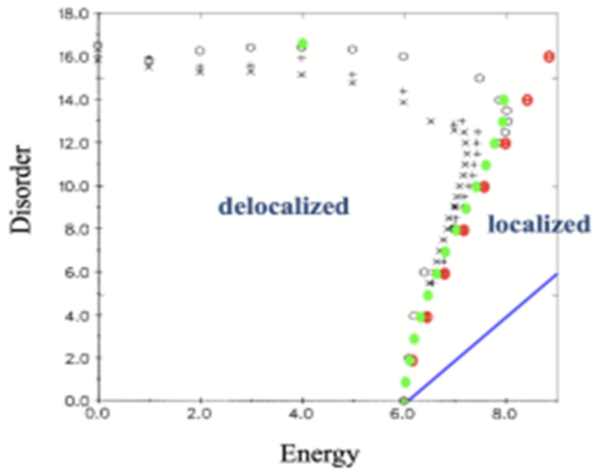


NEE ≡ non-ergodic extended

- Lemarié et al:
  - "Critical properties of the Anderson transition on random graphs: Two-parameter scaling theory, Kosterlitz–Thouless type flow and MBL" (PRB, 2022).
  - "2D Anderson localization and KPZ sub-universality classes: sensitivity to boundary conditions and insensitivity to symmetry classes" (arXiv: 2504.17010).

More Doubts...

- Filoche – Pelletier – Delande – Mayboroda,
  "The Anderson mobility edge
  as a percolation transition",
  arXiv: 2309.03813



- Oliveira – de Lima – Pinheiro – dos Santos,
  "Berezinskii – Kosterlitz – Thouless transition and multifractal critical phase in 2D quantum percolation",
  arXiv: 2510.22148

- I. Horvath,
  "The infrared phase of QCD and Anderson localization",
  arXiv: 2506.04114

Remark. The existence of a fractal phase has been firmly established for the Levy Rosenzweig – Porter model; see Kravtsov et al., SciPost Phys. 18, 010 (2025)
New J. Phys. 17, 12202 (2015)

Outline of Talk

- Introduction: Models, Spectral Types, MP-Results, Physics Picture, Doubts

- Field-theoretical approach:
  - hyperbolic symmetry; orbit structure
  - phase diagram from symmetry breaking: pp, ac, sc

- Wegner model solved in AAT approximation:
  - the model and its Fourier-Laplace transform
  - integro-differential equation from AAT self-consistency
  - novel type of solution with sc-characteristics
  - heuristic picture

Field-theoretical approach: hyperbolic symmetry

Express retarded & advanced Green's functions as Gaussian integrals:

$$G_{E\pm i\varepsilon}(n,n') = \langle n | (E \pm i\varepsilon - H)^{-1} | n' \rangle = \mp i \, \langle \varphi(n)\, \overline{\varphi}(n') \rangle_{\pm} \, ,$$

$$\langle \bullet \rangle_{\pm} = \mathrm{Det}\big(\varepsilon \mp i(E-H)\big) \int_{\varphi,\overline{\varphi}} \bullet \; e^{\pm i \overline{\varphi}\cdot(E \pm i\varepsilon - H)\varphi} \, ,$$

$$\mathrm{Det}(A) = \int_{\psi,\overline{\psi}} e^{\overline{\psi}\cdot A\psi} \quad (\text{"Faddeev–Popov"}) .$$

Observation (Wegner, 1979). The field-theory integrand ("Lagrangian") for, say, $\mathbb{E}\big(G_{E+i\varepsilon}(n,n')\, G_{E-i\varepsilon}(n',n)\big)$, $\varepsilon \longrightarrow 0$, depends only on bilinears $\overline{\varphi}_{+}\varphi_{+} - \overline{\varphi}_{-}\varphi_{-}$ + Grassmann variables.

$\curvearrowright$ The group of (global) symmetries contains $SU(1,1)$.

Field-theoretical approach: orbit structure

Taking the disorder average. Let $H = H_{reg} + H_{ran}$.

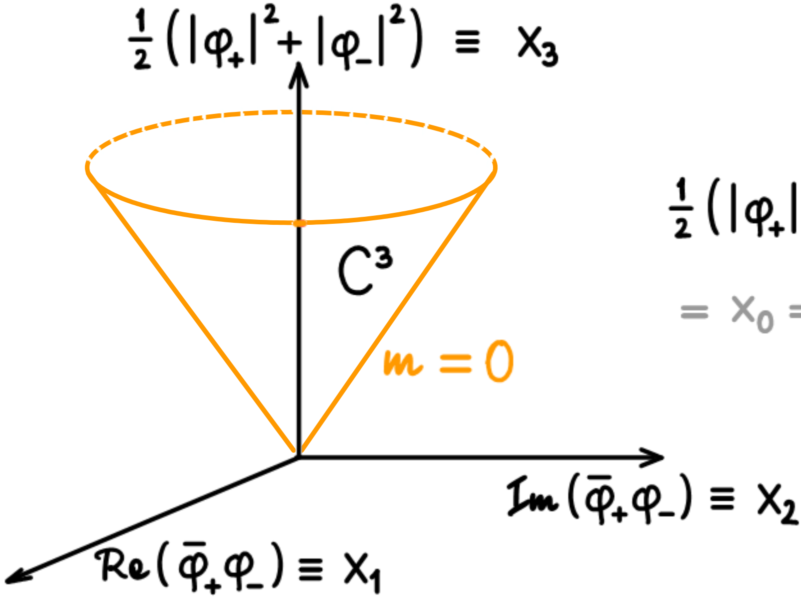
$$\mathbb{E}\left(e^{i\bar{\varphi}\cdot H_{ran}\varphi}\right) = \chi(\varphi\bar{\varphi})$$
$\leftarrow$ characteristic function of the disorder distribution

For a Gaussian random potential (zero mean, variance w_V^2):

$$\mathcal{L}_{bos} = i\bar{\Phi}^\alpha(E-H_{reg})\Phi_\beta(\sigma_3)^\beta{}_\alpha - \tfrac{1}{2}w_V^2\left(\Phi_\alpha(\sigma_3)^\alpha{}_\beta\bar{\Phi}^\beta\right)^2, \quad \Phi = (\varphi_+ \ \ \varphi_-).$$

$(\varepsilon \longrightarrow 0+)$

The 3-dimensional field target space is a convex cone, C^3, foliated by hyperbolic orbits, H^2_m, of the symmetry group $SU(1,1)$.



$\frac{1}{2}(|\varphi_+|^2-|\varphi_-|^2) \equiv m$

$= x_0 = \pm\sqrt{x_3^2-x_1^2-x_2^2}$

Strong-disorder regime (w_V large): dominance of null-orbit.

Phase Diagram from Field-Theory Scenario of Symmetry Breaking

pp-spectrum: unbroken hyperbolic symmetry

In the regime of strong disorder (or low density of states) the global $SU(1,1)$ symmetry (broken by a finite regularization $\varepsilon > 0$) is restored in the limit of $\varepsilon \to 0+$. That is,

field configurations uniformly populate the hyperbolic orbits $H_m^2 \subset \mathbb{C}^3$ (carrying a transitive action of $SU(1,1)$).

Characterization / diagnostic:

$\mathrm{vol}(H_m^2) = \infty$, and field zero mode "delocalized" on H_m^2

$$\Longrightarrow \mathbb{E}\left(\left|G_{E+i\varepsilon}(n,n)\right|^2\right) \sim \frac{1}{\varepsilon} \to \infty$$

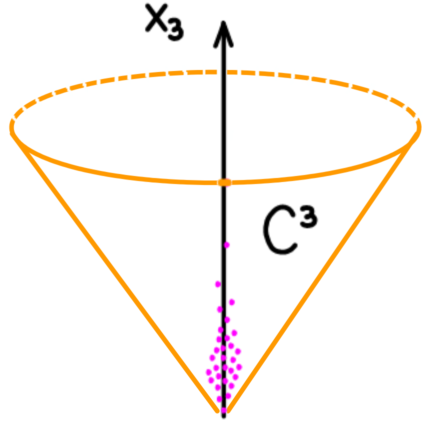
(pp-spectrum ✓)

Remark. In order for symmetry restoration to happen, the presence of the fermionic field degrees of freedom is crucial!

ac-spectrum: spontaneous symmetry breaking

In the regime of weak disorder (or high density of states) the global SU(1,1) symmetry remains spontaneously broken in the limit of turning off ($\varepsilon \to 0$) the symmetry breaking term $+\varepsilon \sum_n x_3(n) > 0$.

As a matter of fact, field configurations cluster in a tubular neighborhood of the bottom part of the x_3-axis of the target cone C^3.



Characterization / diagnostic:

Field zero mode "localized" near bottom of field x_3-axis

$$\Longrightarrow \quad \mathbb{E}\left(\left| G_{E+i\varepsilon}(n,n) \right|^2 \right) \sim \text{const} < \infty . \quad (\text{ac-spectrum} \checkmark)$$

Remark. The physics "proof" proceeds by perturbative renormalization of a nonlinear σ-model for the coarse-grained matrix field $Q = \begin{pmatrix} |\varphi_+|^2 & -\bar{\varphi}_+ \varphi_- \\ \bar{\varphi}_- \varphi_+ & -|\varphi_-|^2 \end{pmatrix}$.

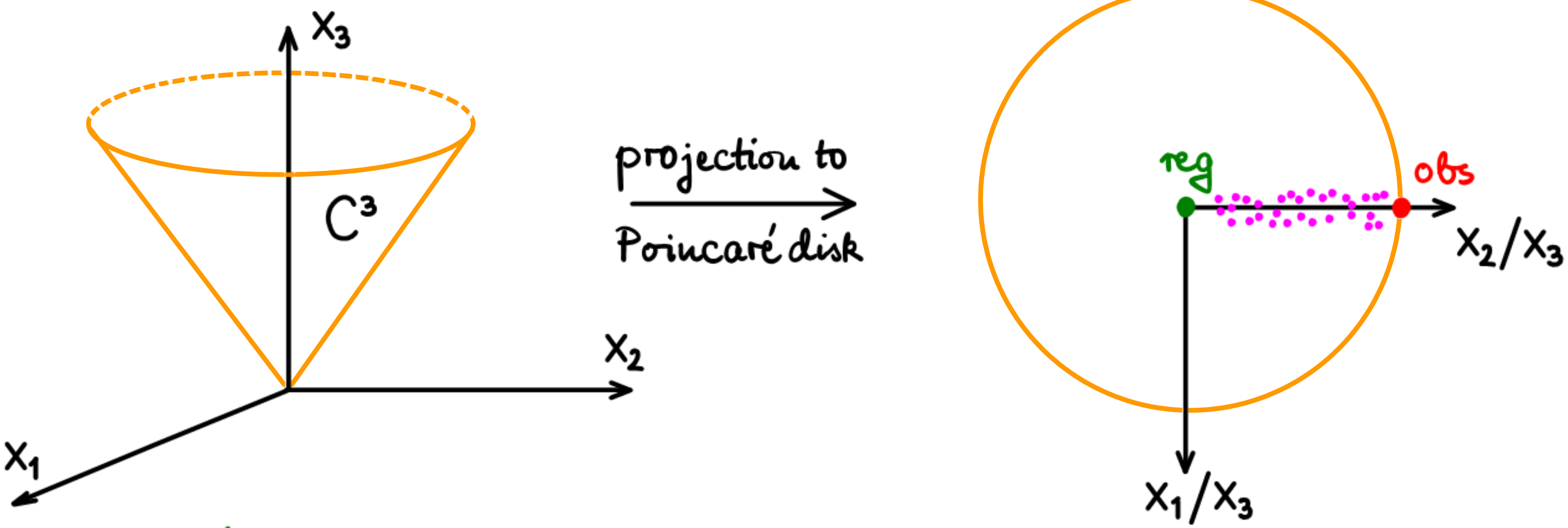
sc-spectrum: novel scenario of "partial" symmetry breaking

For moderately strong disorder, and in high space dimension, we <u>expect</u> an unusual pattern of symmetry breaking. Consider

$$\mathbb{E}\left(|\psi(n)|^{2q}\right) = \left\langle e^{-\varepsilon \sum_{\bullet} X_3(\bullet)} \left(X_3(n) + X_2(n)\right)^q \right\rangle_{F.T.}$$

reg obs

<u>Proposal</u>: field configurations populate a tubular neighborhood of the target-space line that connects the regularization with the observable:



Characterization/diagnostic.

$$\mathbb{E}\left(|G_{E+i\varepsilon}(n,n)|^2\right) \cong \int_0^\infty dt\, e^{-2\varepsilon t}\, \mathrm{Prob}_{n\to n}(t) \sim \varepsilon^{-1+\alpha}$$

(Hölder continuity exponent α).

Wegner model (N=1) solved in the self-consistent approximation of Abou-Chacra, Anderson & Thouless

Wegner Model ($N=1$).

is a random Hamiltonian $H = \sum_{n,n'} h_{n,n'} c_n^+ c_{n'}$ of discrete Schrödinger type
(on graph $\mathbb{G}$; nodes denoted by n),

where $h_{n,n'} = \overline{h_{n',n}}$ Gaussian-distributed complex random variables with mean

$\mathbb{E}(h_{n,n'}) = 0$ and covariance $\mathbb{E}(h_{n,n'} h_{\ell,\ell'}) = \delta_{n,\ell'} \delta_{n',\ell} \left(\delta_{n,n'} w_V^2 + A_{n,n'} w_T^2\right)$.

adjacency matrix of $\mathbb{G}$

H (actually, h) acts on the single-particle Hilbert space $\ell^2(\mathbb{G})$.

Strong-disorder regime: $w_V^2 \gg w_T^2$ (for energy $E=0$).

Note. The probability law of the Wegner model is invariant under local $U(1)$-phase rotations $h_{n,n'} \longmapsto e^{i\theta_n} h_{n,n'} e^{-i\theta_{n'}}$.

Characteristic function Y.

Consider for the resolvent operator $G = (z - h)^{-1}$, $z = E \pm i\varepsilon$,

the 2×2 matrix generalization $\mathcal{G} = (E \cdot 1_2 + i\varepsilon\sigma_3 + \delta\sigma_2 - h)^{-1}$, $|\delta| < \varepsilon$.

– Focus on the diagonal element $\mathcal{G}(n,n) \equiv \mathcal{G}_n$ of the matrix Green's function.

– Introduce the Fourier–Laplace transform of the random matrix $\mathcal{G}_n^{-1}$:

$$Y(Q) = \mathbb{E}\left(e^{i \operatorname{Tr} \mathcal{G}_n^{-1} Q}\right), \quad Q = \begin{pmatrix} |\varphi_+|^2 & -\bar{\varphi}_+ \varphi_- \\ \bar{\varphi}_- \varphi_+ & -|\varphi_-|^2 \end{pmatrix}.$$

Remark.

The switch from the basic vector variables $\varphi_\pm$ to the matrix variable Q is possible thanks to the Wegner model's invariance under local $U(1)$-phase rotations.

Self-consistent approximation

(Abou-Chacra, Anderson, Thouless 1973)

By basic linear algebra, our local matrix Green's function satisfies the relation

$$\mathcal{G}_n^{-1} = (E - h_{n,n}) \cdot 1_2 + i\varepsilon\sigma_3 + \delta\sigma_2 - \sum_{\substack{n_1 n_2 \\ (\neq n)}} h_{n,n_1} \mathcal{G}'_{n_1,n_2} h_{n_2,n}$$

Green's fctn on $\mathbb{G}$ with node n removed.

AAT suggested the following closed equation for (the law of) the random matrix $\mathcal{G}_n$:

$$\mathcal{G}_n^{-1} \underset{\text{in distribution}}{\overset{!}{=}} (E - h_{n,n}) \cdot 1_2 + i\varepsilon\sigma_3 + \delta\sigma_2 - \sum_{n' (\neq n)} |h_{n,n'}|^2 \mathcal{G}_{n'}$$

sum over K terms (when coordination no. $= K+1$).

Remark. The AAT approximation is exact on a tree graph ("Bethe lattice").

It is expected to be reasonable in high space dimension.

Integro-differential equation (AAT) for Y.

The AAT approximation translates to a self-consistency equation for our Fourier–Laplace transform $Y(Q)$:

$$Y(Q) = e^{i \operatorname{Tr} Q (E + i\varepsilon\sigma_3 + \delta\sigma_2) - \frac{1}{2} w_V^2 \operatorname{Tr} Q^2} \times \left(\int_{C^3} d\mu(Q')\, e^{-w_T^2 \operatorname{Tr} Q Q'} (\Delta Y)(Q') \right)^K, \text{ where}$$

Δ denotes the hyperbolic Laplacian $\frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_0^2}$ restricted to the cone C^3.

[Recall $Q = x_0 \cdot 1_2 + x_1 i\sigma_1 + x_2 i\sigma_2 + x_3 \sigma_3$

and $C^3 : x_3^2 - x_2^2 - x_1^2 - x_0^2 = 0$.]

– Adopt horospherical coordinates:

$x_0 = \frac{1}{2}(|\varphi_+|^2 - |\varphi_-|^2) \equiv m$, $\quad x_1 = \operatorname{Re}(\bar{\varphi}_+ \varphi_-) \equiv b$,

$x_3 = \frac{1}{2}(|\varphi_+|^2 + |\varphi_-|^2) = \frac{1}{2}(e^{\phi} + (m^2+b^2)e^{-\phi})$, $\quad x_2 = \operatorname{Im}(\bar{\varphi}_+\varphi_-) = \frac{1}{2}(e^{\phi} - (m^2+b^2)e^{-\phi})$.

Then, $d\mu(Q) = \pi^{-1} dm\, d\phi\, db$.

Novel type of solution with sc-character.

Let the energy parameter $E = 0$, for simplicity. It is then easy to see that the AAT equation for Y has pp-type solutions for strong disorder and ac-type solutions for weak disorder ($w^2 \equiv w_V^2 / w_T^2 \ll K^2$).

— By SU(1,1)-harmonic analysis one ascertains that the pp-solution becomes unstable for $w < w_c$ where $K = \frac{1}{\sqrt{2}} \frac{w_c}{\ln w_c}$ (critical disorder w_c for $E = 0$).

• For good analytical control, let the dimension parameter K (hence w_c) be large. We are then able to demonstrate that

$$Y_{sc} \equiv e^{-2w_V^2 m^2 - 2\varepsilon e^{\phi} - \beta\left((m^2+b^2)\, e^{-\phi}\right)^{1/2}}$$

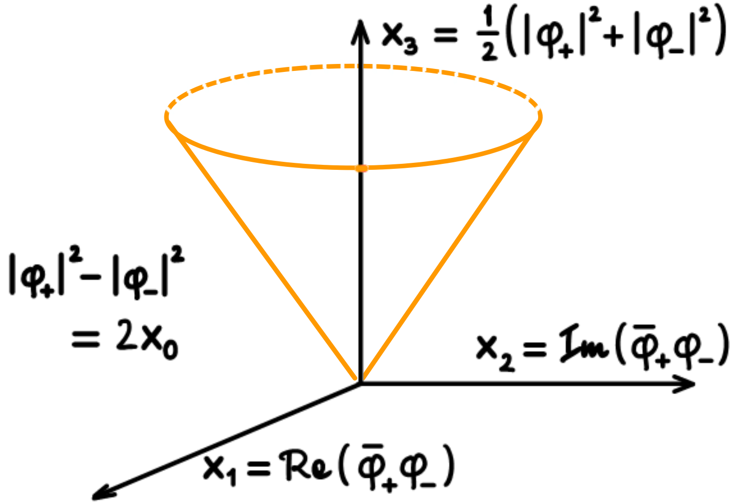
is a solution for $w = w_c$ and $\varepsilon = \delta \longrightarrow 0+$.

Spectral diagnostic. $\mathbb{E}\left(|G_{0+i\varepsilon}(n,n)|^2\right) = \int Y_{sc}(Q)\, d\mu(Q) \sim \frac{1}{\sqrt{\varepsilon}}$,

i.e. singular continuous with Hölder exponent $\alpha = \frac{1}{2}$.

Argument for novel scenario of symmetry breaking.

— By the factor $e^{-\frac{1}{2}w_V^2 \mathrm{Tr}\, Q^2} = e^{-2w_V^2 x_0^2}$, strong disorder $w_V^2 \gg w_T^2$ constrains the field to stay close to the null surface $x_0 = 0$.



Constraint from $\mathrm{Det}(Q) = 0$:

$$x_3 = +\sqrt{x_0^2 + x_1^2 + x_2^2}\,.$$

- For $x_0 \to 0$, the range of the integral kernel

$$e^{-w_T^2 \mathrm{Tr}\, Q' Q} = e^{2w_T^2(-x_0' x_0 - x_3' x_3 + x_1' x_1 + x_2' x_2)}$$ is

(i) infinite on time-like geodesics $Q \longmapsto e^{\varphi\sigma_1} Q\, e^{-\varphi\sigma_1}$,

(ii) finite on space-like geodesics $Q \longmapsto e^{i\theta\sigma_3} Q\, e^{-i\theta\sigma_3}$.

Thank you!