

Quantum mixing for Large Schreier Graphs

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Context and earlier results

Wavefunction delocalization

Setting: Infinite graph G which is spectrally delocalized: it has absolutely continuous spectrum in some I .

Problem: Deduce the wavefunctions are delocalized ??

Plan: Approximate G by finite graphs $G_N \xrightarrow{BS} G$. Show that any o.n.b. $(\psi_j^{(N)})$ of G_N is uniformly distributed on G_N (QE) and analyze correlations $\overline{\psi_j^{(N)}(x)\psi_k^{(N)}(y)}$ (QM).

Previous results (Anantharaman-S. 17-19)

Here K is an observable. So for most eigenfunctions

$$\sum_{x,y \in G_N} \overline{\psi_\alpha(x)} K(x,y) \psi_\alpha(y) \approx \sum_{x,y \in G_N} K(x,y) \Phi_\alpha(x,y)$$

Here $\Phi_\alpha(x,y) \approx \frac{1}{N} \frac{\Im R^{\lambda_\alpha}(x,y)}{\Im R^{\lambda_\alpha}(x,x)}$. In sense of measures on G_N ,

$$|\psi_\alpha^{(N)}(x)|^2 \approx \frac{1}{N}.$$

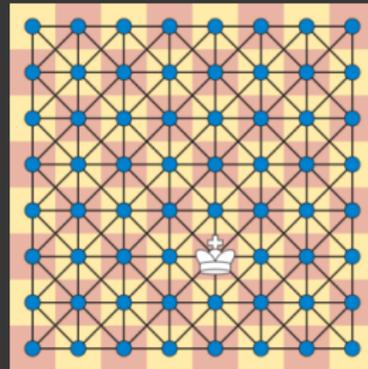
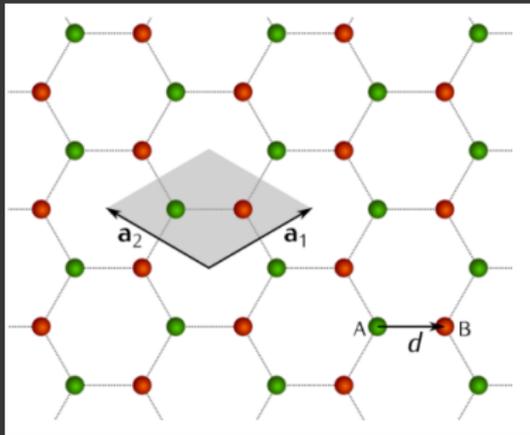
More precisely $\sum_{x \in \Lambda_N} |\psi_\alpha(x)|^2 \approx c$ if $|\Lambda_N| = cN$, $0 < c < 1$.

Applies to the **Anderson model** on such trees.

Earlier for the adjacency matrix on regular $G_N \xrightarrow{BS} \mathbb{T}_d$
(Anantharaman-Le Masson-Brooks-Lindenstrauss).

Previous results (McKenzie-S. 22-23)

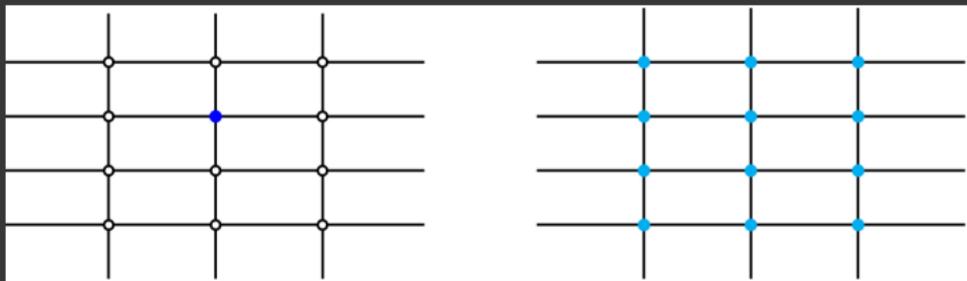
QE (with caveats) to \mathbb{Z}^d -periodic graphs with no flat bands.



Previous results

(Magee-Thomas 22, Naor-Sah-Sawhney-Zhao 22, Letrouit-Burq 24): For Cayley graphs (G_N) of quasi-random groups (and more), with high probability, a random orthonormal basis is quantum ergodic.

There are **dynamical versions** of quantum ergodicity on \mathbb{Z}^d -periodic graphs (Boutet de Monvel-S. 24).



Problem

(Bauerschmidt-Huang-Yau 16+): Probabilistic QUE for the adjacency matrix of random regular graphs. (These converge to \mathbb{T}_d).

- ▶ What about graphs G which are not trees and not \mathbb{Z}^d -periodic ?
- ▶ What about the measure $\overline{\psi_j^{(N)}(x)\psi_k^{(N)}(x)}$?

We give some answers when G is a **Cayley graph** and $G_N \xrightarrow{BS} G$ are **Schreier graphs**. These graphs are regular. However, we give extensions to irregular graphs.

Quantum mixing on manifolds

Thm: (Zelditch, Sunada) Let (M, g) be a compact Riem. manifold and (λ_j, ψ_j) be an o.n.b. of $-\Delta_M$.

Let A be a zero order Ψ DO and $\omega(A)$ the average of σ_A over the unit cotangent bundle. Consider properties

$$(i) \lim_{\lambda \rightarrow \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle \psi_j, A\psi_j \rangle - \omega(A)|^2 = 0.$$

$$(ii) \lim_{\delta \downarrow 0} \limsup_{\lambda \rightarrow \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j < \lambda} \sum_{\substack{k \neq j, \lambda_k \leq \lambda \\ |\lambda_j - \lambda_k| < \delta}} |\langle \psi_j, A\psi_k \rangle|^2 = 0.$$

$$(iii) \text{ For any } \tau \in \mathbb{R}, \\ \lim_{\delta \downarrow 0} \limsup_{\lambda \rightarrow \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j < \lambda} \sum_{\substack{k \neq j, \lambda_k \leq \lambda \\ |\lambda_j - \lambda_k - \tau| < \delta}} |\langle \psi_j, A\psi_k \rangle|^2 = 0.$$

Eigenstate Thermalization

Then the geodesic flow on $(S^*M, d\mu)$ is ergodic iff (i) and (ii) hold. It is weakly mixing iff (i), (ii) and (iii) hold.

For **Wigner matrices**, (Cipolloni-Erdős-Schröder 21):

$$|\langle u_i, Au_j \rangle - \delta_{i,j} \langle A \rangle| \leq \frac{N^\xi \|\mathring{A}\|}{\sqrt{N}} \quad w.h.p.$$

\forall eigenvectors u_i and u_j , where A is a matrix observable, $\langle A \rangle = \frac{1}{N} \text{tr}(A)$ and $\mathring{A} = A - \langle A \rangle$. They called this *eigenstate thermalization*. $\|\mathring{A}\|$ was improved later.

Also for **deformed Wigner-type** matrices (+Henheik 24) and **band matrices** in $d = 1$, down to the optimal delocalized regime (Yau-Yin 25).

Our results with Charles and Cyril

Some Cayley graphs we have in mind

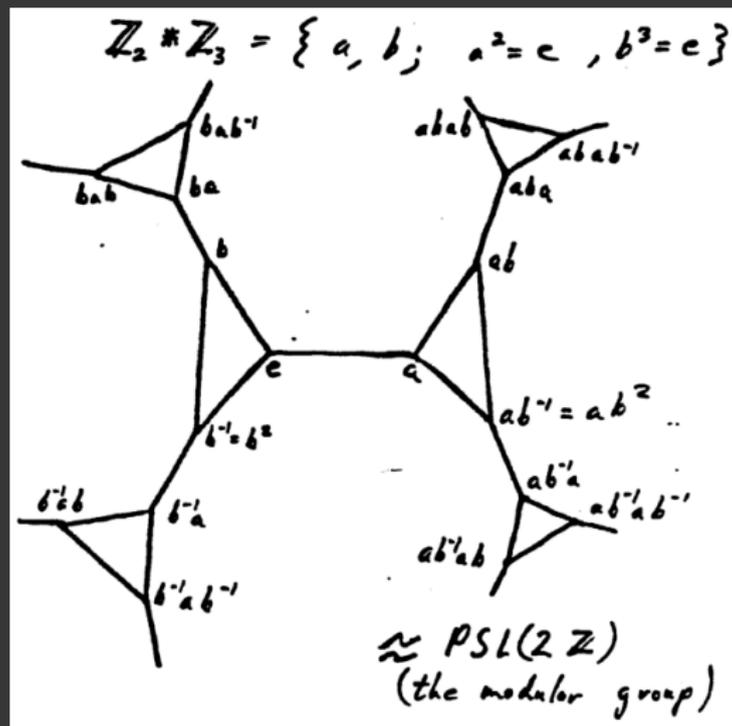
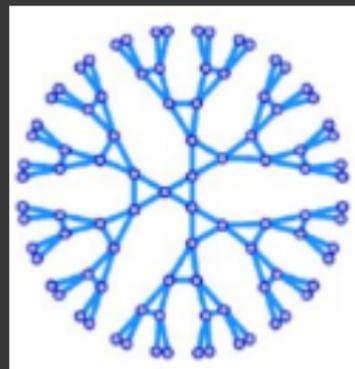
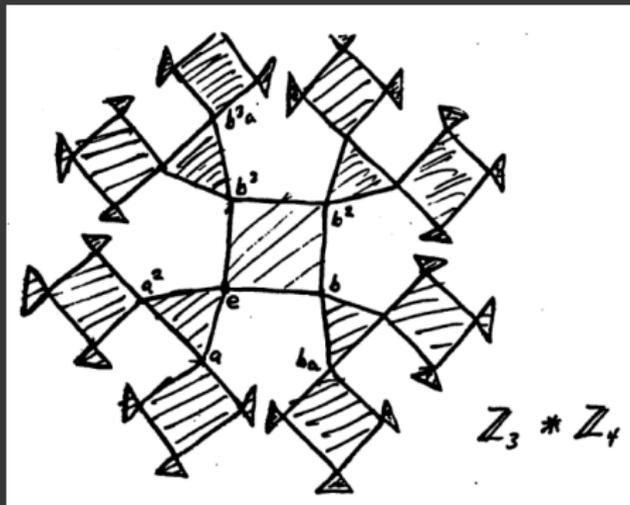


Figure: From McLaughlin thesis, 1987

Some Cayley graphs we have in mind



Quantum mixing on graphs

If $G_N \xrightarrow{BS} G$, our aim is to prove an estimate of the form

$$\lim_{\eta \downarrow 0} \lim_{N \rightarrow \infty} \frac{1}{N(I)} \sum_{\lambda_\alpha \in I} \sum_{\substack{\lambda_\beta \in I \\ |\lambda_\beta - \lambda_\alpha - \tau| < \eta}} |\langle \psi_\alpha, K \psi_\beta \rangle - \delta_{\alpha, \beta} \langle K \rangle_\alpha|^2 = 0$$

where $\psi_\alpha = \psi_\alpha^{(N)}$ is an orthonormal eigenbasis of G_N .

We want to prove this for Cayley graphs G which are spectrally delocalized.

A remark on observables

Before stating the results, let us remark this example:

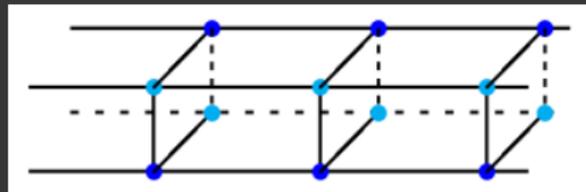


Figure: \exists o.n.b. $(\psi_\alpha)_{\alpha=1,\dots,4N}$ with N elements supported on the dark vertices, and N elements supp. on the light ones. They are still delocalized in the sense that $|\psi_\alpha(x)|^2 = \frac{1}{2N}$. Transversally constant observables don't see these details. They see uniformly distributed eigenfunctions, i.e.

$$\langle \psi_\alpha, K \psi_\alpha \rangle = \langle K \rangle = \frac{1}{4N} \sum_x K(x).$$

QE/QM for subclasses of observables gives a “low resolution” comparison.

Cayley graphs, Schreier graphs

We assume $G = \text{Cay}(\Gamma, S)$ is a Cayley graph with a symmetric set S of generators. Let $\lambda(g)\delta_v = \delta_{gv}$ be the regular representation. G has vertices Γ and edges (g, sg) , $s \in S$.

Let $\rho_N \in \text{Hom}(\Gamma, S_N)$ be a permutation representation. A Schreier graph $G_N = \text{Sch}(\Gamma, S, \rho_N)$ has vertex set $\{1, \dots, N\}$ and edges $(x, s.x)$, $s \in S$, where $s.x := \rho_N(s)(x)$.

The adjacency matrices are thus

$$\lambda(\mathbf{1}_S) = \sum_{s \in S} \lambda(s) \quad \text{and} \quad \rho_N(\mathbf{1}_S) = \sum_{s \in S} \rho_N(s)$$

The results

We consider more generally weighted versions

$$P = \lambda(p) = \sum_{s \in S} p_s \lambda(s) \quad \text{and} \quad P_N = \rho_N(p) = \sum_{s \in S} p_s \rho_N(s)$$

Here $p_s \in \mathbb{C}$ and $p_{s^{-1}} = \overline{p_s}$, so P is self-adjoint.

Any $2d$ -regular graph is Schreier. Any d -regular is Schreier up to a double cover. Cayley are “sparse”.

We assume P has AC spectrum in I , more precisely,

$$\text{(AC)} \quad \mu_p(I) > 0 \quad \text{and} \quad \sup_{\substack{E \in I \\ \eta > 0}} \Im R^{E+i\eta}(e, e) \leq C_0$$

where $\mu_p(I) = \langle \delta_e, \chi_I(P) \delta_e \rangle$ is the spectral measure of P and $R^z = (P - zI)^{-1}$ is the resolvent.

Main results (Bordenave-Letrouit-S.)

Let us consider observables of the form

$$K_N(x, y) = k_N(x) \mathbf{1}(x = t.y) \quad \text{for some } t \in \Gamma.$$

If $t = e$ then K_N is diagonal. We define the averages

$$\langle k_N \rangle = \frac{1}{N} \sum_{x \in G_N} k_N(x) \quad \text{and} \quad \langle K_N \rangle_{\phi, \psi} = \langle k_N \rangle \langle \phi(t.\cdot), \psi \rangle$$

Then our aim is to prove *quantum mixing (QM)*:

$$\lim_{\eta \downarrow 0} \lim_{N \rightarrow \infty} \frac{1}{N(I)} \sum_{\lambda_\alpha \in I} \sum_{\substack{\lambda_\beta \in I \\ |\lambda_\beta - \lambda_\alpha - \tau| < \eta}} |\langle \psi_\alpha, K \psi_\beta \rangle - \langle K \rangle_{\psi_\alpha, \psi_\beta}|^2 = 0.$$

Roughly speaking, $\overline{\psi_\alpha(t.x)} \psi_\beta(x) \approx \frac{1}{N} \langle \psi_\alpha(t.\cdot), \psi_\beta \rangle$.

Main results (Bordenave-Letrouit-S.)

In the special case of $G = \mathbb{T}_q$ the regular tree, one should have $\overline{\psi_\alpha(t.x)}\psi_\alpha(x) \approx \frac{1}{N} \frac{\Im R^{\lambda\alpha}(e,t)}{\Im R^{\lambda\alpha}(e,e)}$.

Define the *asymptotic covariance of K* by

$$\sigma(g) = \limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{x \in G_N} (k_N(x) - \langle k_N \rangle)(k_N(g.x) - \langle k_N \rangle) \right|$$

Thm 1: Let G be a Cayley graph satisfying **(AC)**, let $G_N \xrightarrow{BS} G$ be Schreier graphs. Suppose K_N satisfy $\|k_N\|_\infty \leq 1$. If K_N are *asymptotically uncorrelated*, i.e. $\lim_{|g| \rightarrow \infty} \sigma(g) = 0$, then QM holds.

More assumptions: RD property

Lem: If $\{k_N(x)\}_{x \in G_N}$ are i.i.d. then they are uncorrelated almost surely.

For the next theorems, we need more assumptions:

(RD) The group Γ satisfies the rapid decay property: there is a set S of generators of Γ such that $\forall p \in \mathbb{C}[\Gamma]$,

$$\|\lambda(p)\|_{op} \leq C (\text{diam}_S(p))^{C_1} \|p\|_2,$$

Examples: Free groups, groups with polynomial growth (e.g. \mathbb{Z}^d), hyperbolic groups, Cartesian and free products of such groups, $SL_2(\mathbb{Z})$ but not $SL_n(\mathbb{Z})$, $n \geq 3$.

More assumptions: off-diagonal decay of R^Z

(MD) The resolvent has the moment decay:

$$\limsup_{\eta \downarrow 0} \sum_{E \in I} \sum_{g \in \Gamma} \eta^2 |\Im R^{E+i\eta}(e, g)|^4 |g|^{2C_1+2} = 0.$$

Note: **(AC)** and Ward's identity already imply that

$$\limsup_{\eta \downarrow 0} \sum_{E \in I} \sum_{g \in \Gamma} \eta^2 |\Im R^{E+i\eta}(e, g)|^4 = 0.$$

Thm 2: Let G be a Cayley graph with **(AC)**, **(RD)** and **(MD)**. Let $G_N \xrightarrow{BS} G$. Fix a finite $T \subset \Gamma$. Then $\exists H_N \subset \mathbb{C}^N$ with $\dim H_N = o(N)$ such that for any T -local matrix (K_N) with $\|K_N\|_\infty \leq 1$, if $K_N \in H_N^\perp$, then quantum mixing holds.

Main results (Bordenave-Letrouit-S.)

H_N is spanned by eigenvectors of operators $Q_N \approx Q(P_N)$ outside the spectrum of $2Q$.

To strengthen the conclusion we add a last assumption.

(SC) $G_N \xrightarrow{BS} G$ and for any $p \in \mathbb{C}[\Gamma]$,

$$\lim_{N \rightarrow \infty} \|\rho_N(p)_{\mathbf{1}^\perp}\|_{op} = \|\lambda(p)\|_{op}.$$

Thm 3: Let G be a Cayley graph with **(AC)**, **(RD)** and **(MD)**. Suppose (G_N) satisfies **(SC)**. Then for **any** T -local matrix (K_N) with $\|K_N\|_\infty \leq 1$, QM holds.

Examples

Property	(AC)	(MD)	(SC)
Free products	Yes	Yes	Yes but not written...
\mathbb{Z}^d	Yes	No	Yes
Surface groups	?	?	Yes
Some RACG $\times \mathbb{Z}^2$	Yes	Yes	Yes

The thms hold more generally for matrix-valued p_g :

$$A(\mathbf{p}) = \sum_{g \in \Gamma} \mathbf{p}(g) \otimes \lambda(g),$$

where $\mathbf{p}(g) \in M_r$.

Application: Quantum mixing for lifts of non-regular graphs converging to the universal cover.

Proof Ideas

We prove **small-scale mixing** which implies QM:

$$\lim_{\eta \downarrow 0} \limsup_{N \rightarrow \infty} \frac{1}{N\eta} \sum_{\lambda_\alpha \in I_{E_1}^\eta} \sum_{\lambda_\beta \in I_{E_2}^\eta} |\langle \psi_\alpha, K_N \psi_\beta \rangle - \langle K_N \rangle_{\psi_\alpha, \psi_\beta}|^2 = 0$$

where $I_E^\eta = [\lambda_E - \eta, \lambda_E + \eta]$. Let $\langle k_N \rangle = 0$.

1.

$$V_{E_1, E_2} \leq \frac{4\eta}{N} \text{Tr}(\mathfrak{I}R_N^{z_1} K_N \mathfrak{I}R_N^{z_2} K_N^*).$$

This is a **spectral step** using the eigenvalue equation ($\mathfrak{I}R_N^{z_i} \psi_\alpha = \frac{\eta}{|\lambda_\alpha - z_i|^2} \psi_\alpha$ and $\frac{\eta}{|\lambda_\alpha - z_i|^2} \geq \frac{1}{2\eta}$), + a nice observation.

Proof Ideas

2.

$$\frac{\eta}{N} \text{Tr}(\mathfrak{J}R_N^{z_1} K \mathfrak{J}R_N^{z_2} K^*) \approx \frac{\eta}{N} \langle K, f_{E_1, E_2}(P_N) K \rangle.$$

If $E_1 = E_2$ and K is diagonal,

$$(f_{E_1, E_1}(P_N) K)(x) = \sum_{y \in G_N} (\mathfrak{J}R_N^{z_1}(x, y))^2 k(y).$$

This is the **propagator** we need to understand.

3. **For non-correlated observables:** Let

$$\sigma_N(g) = \frac{1}{N} \sum_{x \in G_N} \overline{k_N(x)} k_N(g.x). \text{ Then}$$

$$\begin{aligned} \frac{\eta}{N} \langle K, f_{E_1, E_2}(P_N) K \rangle &\lesssim \eta \sum_{g \in \Gamma} \sigma(g) (\mathfrak{J}R^{z_0}(x, gx))^2 \\ &\leq \eta \sum_{g \in B_S} (\mathfrak{J}R^{z_0}(x, gx))^2 + \delta \eta \sum_g (R^{z_0}(x, gx))^2 \xrightarrow{\eta, \delta} 0. \end{aligned}$$

Proof Ideas

4. In case of **(MD)**+**(SC)**:

$$\begin{aligned}\frac{\eta}{N} \langle K_N, f_{E_1, E_2}(P_N) K_N \rangle &\leq \frac{\eta}{N} \|K_N\|^2 \|f_{E_1, E_2}(P_N)|_{\mathbf{1}^\perp}\|_{op} \\ &\leq c\eta \|f_{E_1, E_2}(P)\|_{op}.\end{aligned}$$

Using **(RD)**, we have

$$\|f_{E_1, E_2}(P)\|_{op} \leq C \max_{E_1, E_2} \left(\sum_g (\Im R^{z_i}(e, g))^4 |g|^{C'_1} \right)^{1/2}.$$

5. With just **(MD)**: Replace $K_N \perp \mathbf{1}$ by $K_N \perp \text{span}\{\chi_{\sigma(2f_{E_1, E_2}(P))^c}(f_{E_1, E_2}(P_N))\}$.

On the necessity of assumptions

(AC) is essentially a minimal assumption to expect wavefunction delocalization.

1. **(AC)** alone cannot control *all* observables. e.g.:
 $\Gamma = \mathbb{Z}$, $\psi_\alpha(x) = e^{2\pi i \alpha x / N}$ and $k_N = e^{2\pi i x / N}$. Then
 $|\langle \psi_\alpha, k_N \psi_{\alpha+1} \rangle|^2 \geq 1$. Another example in [McKenzie](#).
2. In \mathbb{Z}^d all assumptions are satisfied except possibly **(MD)**. QM fails for certain K_N . So **(MD)** fails.
3. Tensor product of \mathbb{Z} with butterfly shows necessity of **(MD)** in matricial case.
4. Cartesian product of \mathbb{Z} with 4-cycle shows necessity of taking $K_N \in \mathbf{L}_r^\perp$ in matricial case.

Outlook

- ▶ Anderson model $(G_N, \omega) \xrightarrow{BS} (\mathcal{T}, \omega)$ with this approach ? Actually

$$\frac{\eta}{N} \langle K_N, f_{E_1, E_1}(P_N) K_N \rangle \approx \frac{\eta}{N} \sum_{n \geq 0} \langle K_N^{z_1}, \mathcal{S}_{z_1}^n \tilde{K}_N^{z_1} + \mathcal{S}_{u^{z_1}}^n \hat{K}_N^{z_1} \rangle_{\mu^{z_1}}$$

issue with error bounds for far-propagation.

- ▶ Free products with just spectral gap ?
- ▶ Better understanding of $\langle K \rangle_{\psi_\alpha, \psi_\beta}$ in some G ?
- ▶ The approach seems promising for manifolds.
- ▶ Case of hyperbolic manifolds being finalized by Kai Hippi (different method).

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