L^∞ norm of chaotic eigenfunctions

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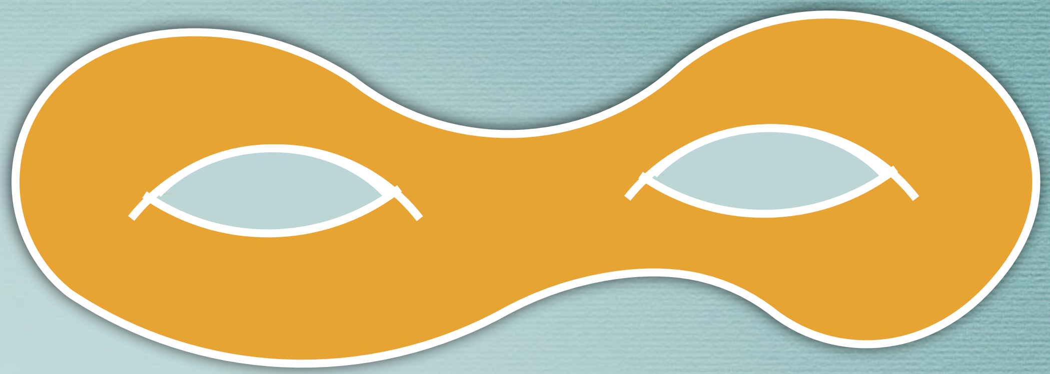
Joint works with Martin Vogel (Strasbourg) and Yann Chaubet (Nantes)

I. Introduction to Quantum Chaos

Quantum chaos



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(ψ_j) orthonormal basis of $L^2(X)$ made of eigenfunctions of the Laplacian:

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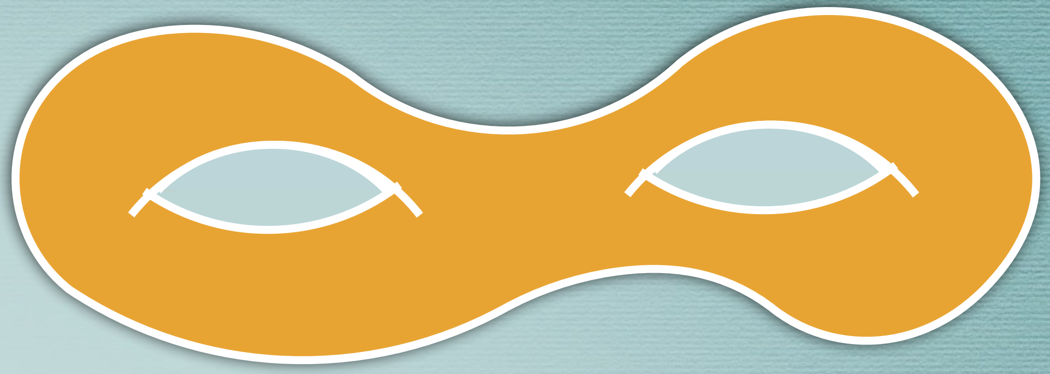

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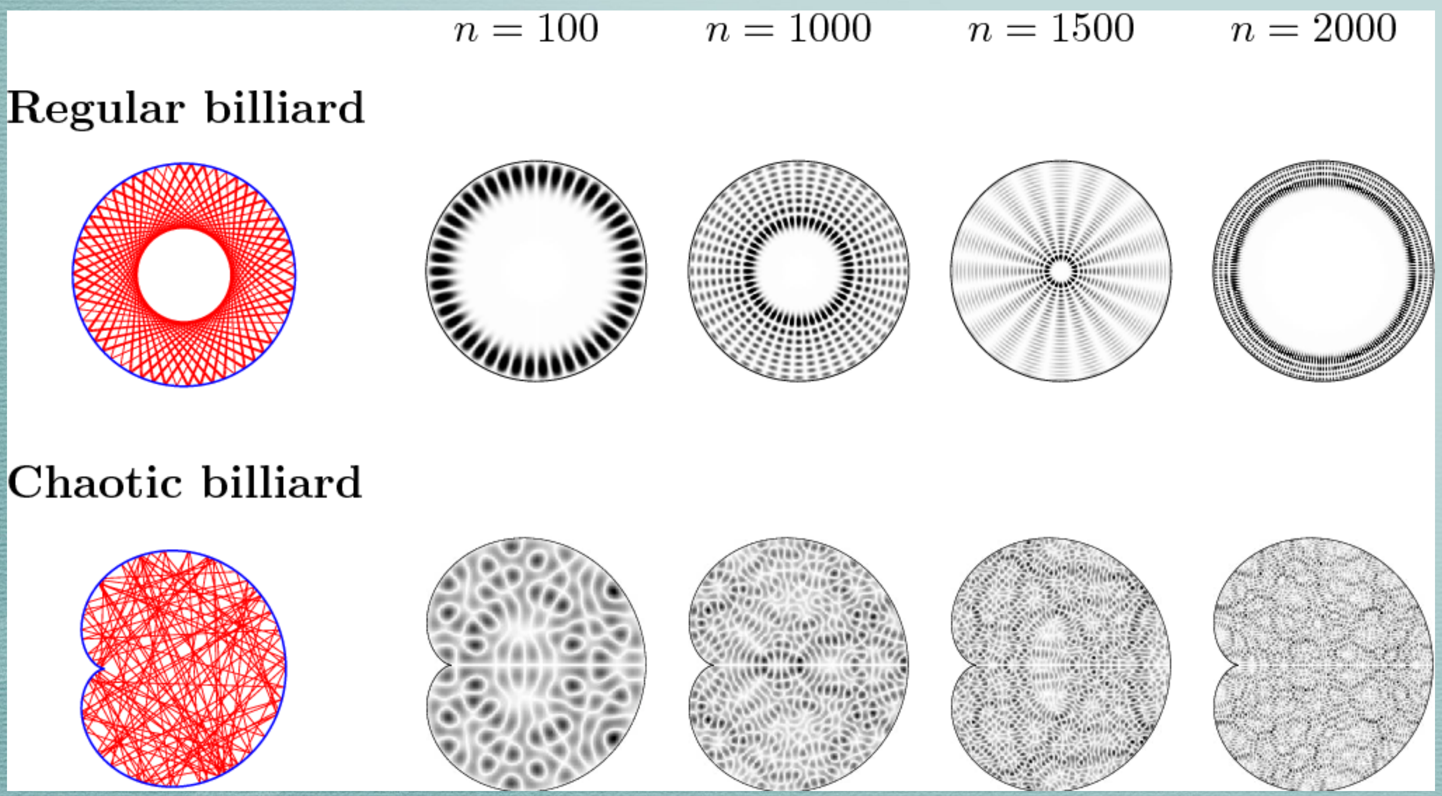

Picture taken from A. Backer, 2007

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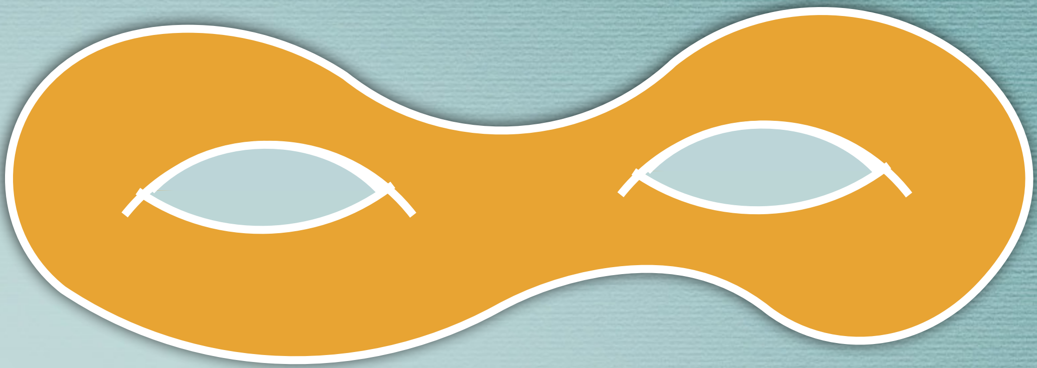

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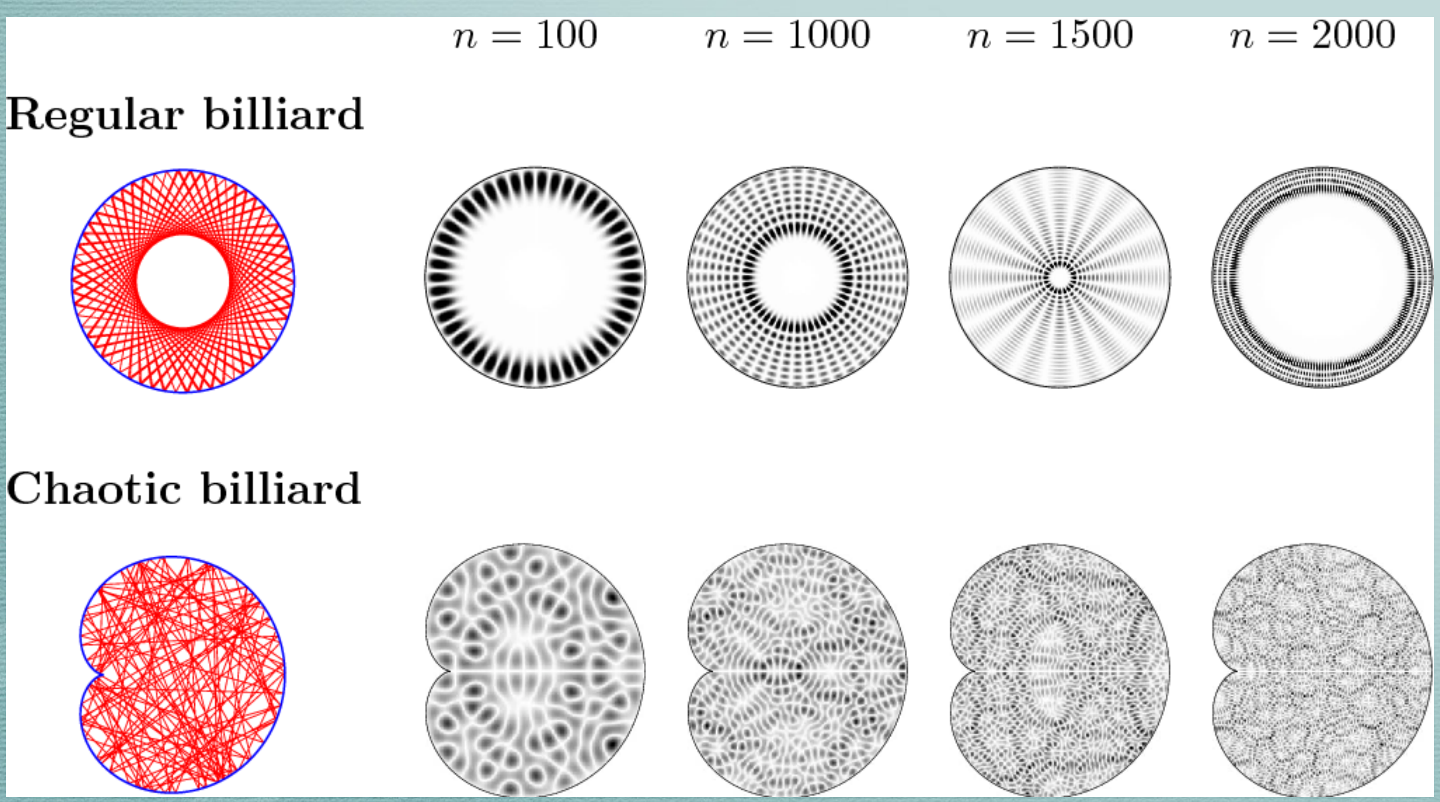

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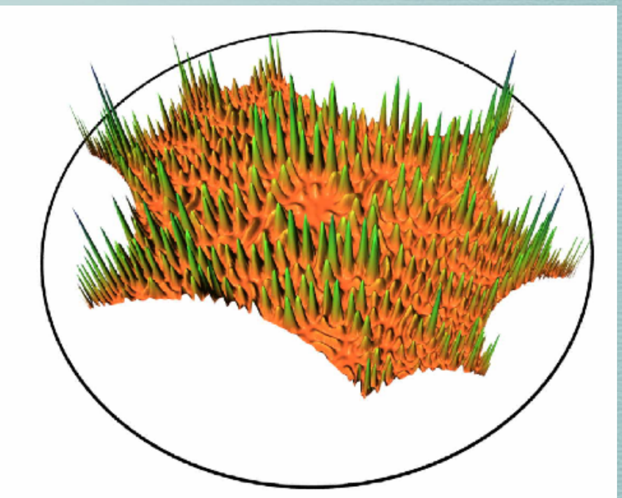

Eigenfunction on a hyperbolic surface (Aurich-Steiner 1992)

Spherical harmonic

Quantum chaos

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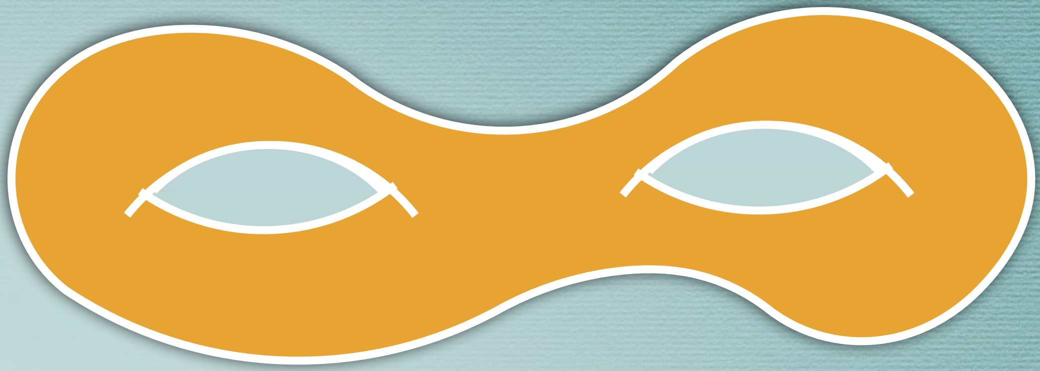

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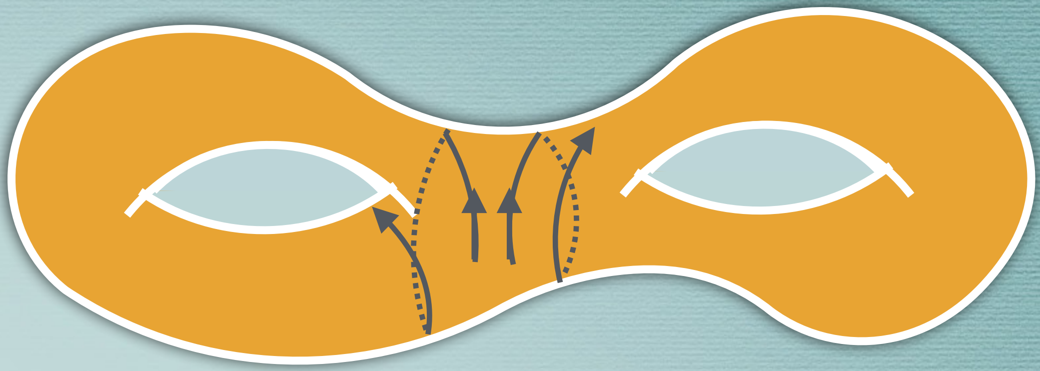

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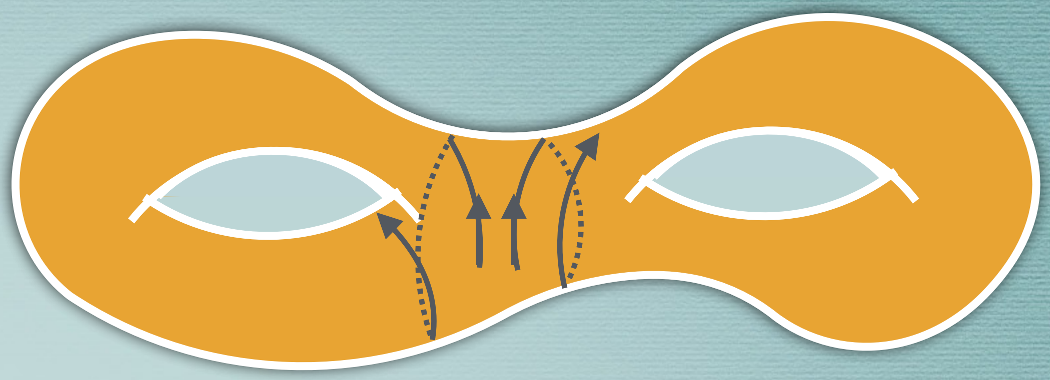

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- $|\psi_j(x)|^2\,\mathrm{d}x$ should converge weakly to the uniform measure (Quantum Unique Ergodicity conjecture)

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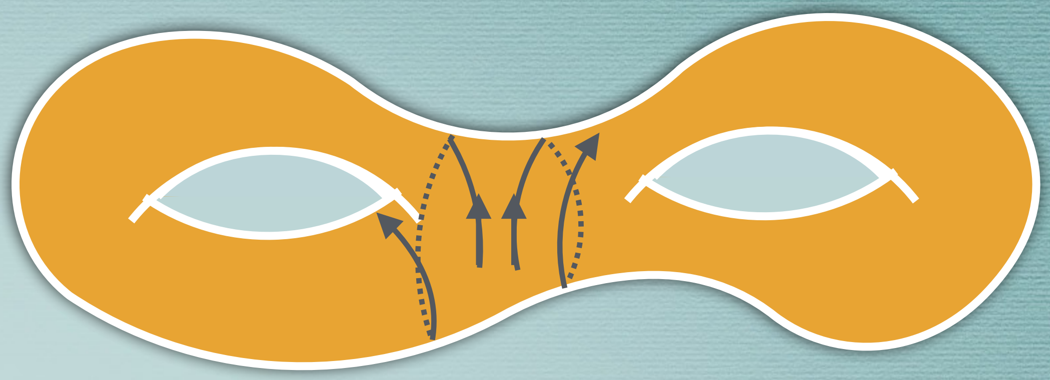

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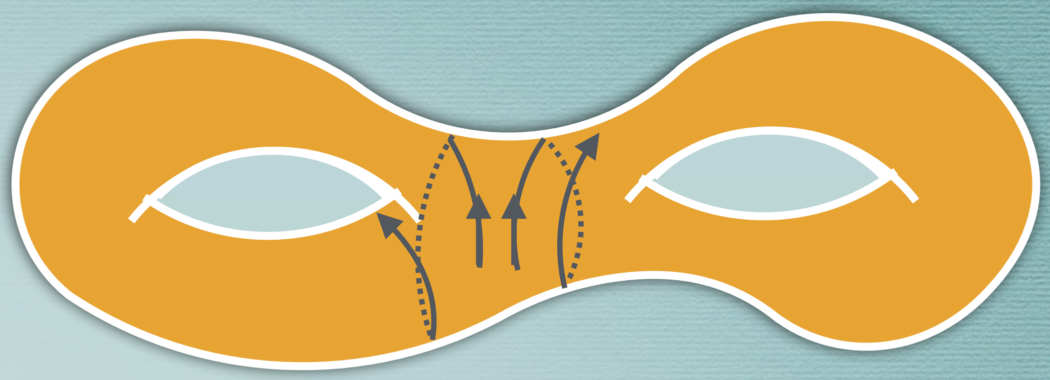

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- $\|\psi_j\|_{L^\infty}$ should not be too large.

II. L^∞ norms of eigenfunctions

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1) Previous bounds on L^∞ norms of eigenfunctions

Existing results on L^∞ norms of eigenfunctions (1)

Theorem (*Avakumovic,Hörmander, Levitan '50-'60*) :
(X, g) compact Riemannian manifold:

$$\left(-\Delta\psi = \lambda\psi\right) \Longrightarrow \|\psi\|_{L^\infty} \leq C\lambda^{\frac{d-1}{4}}\|\psi\|_{L^2}.$$

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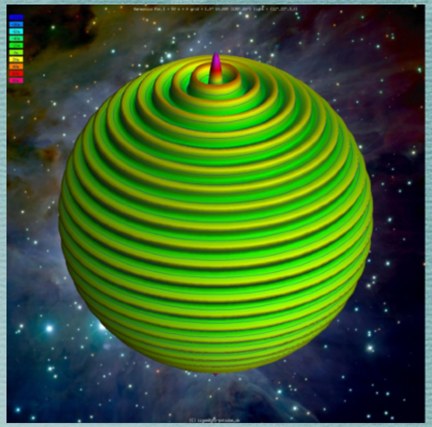

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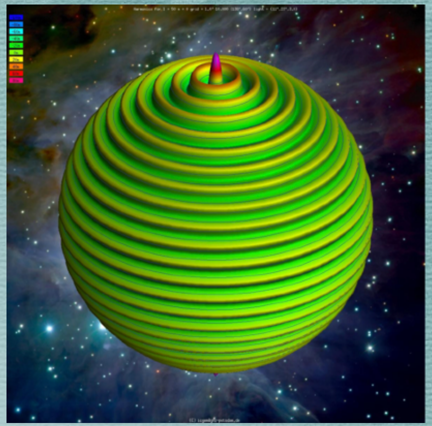

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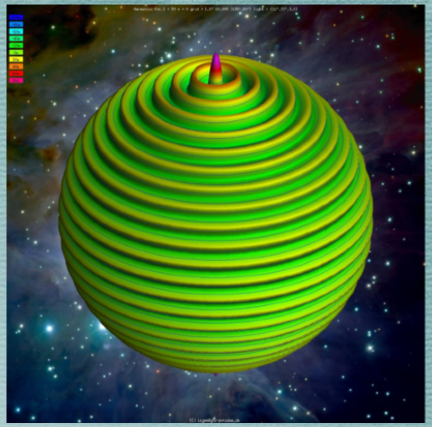

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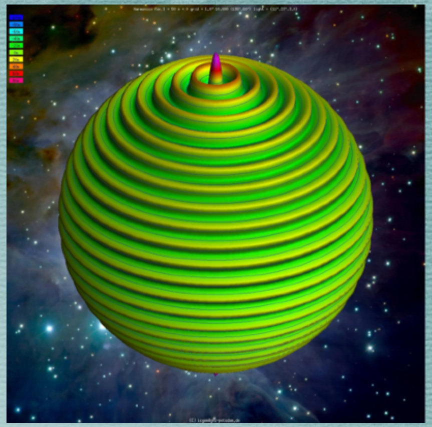

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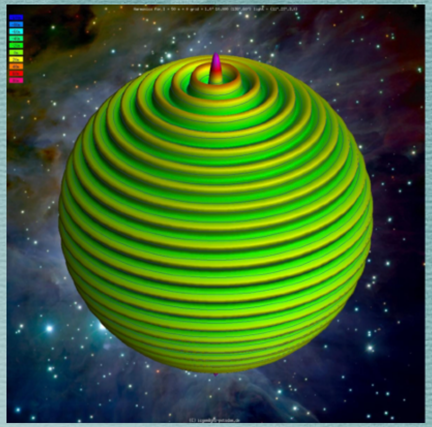

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Recent generalizations to other L^p norms and other manifolds: Hassell-Tacy '15, Hezari-Rivière '16, Bonthonneau '17, Blair-Sogge '19, Canzani-Galkowski '23...

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Existing lower bounds on L^∞ norms of eigenfunctions

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Theorem (*Rudnick-Sarnak '94, Milicevic '10, '11*):
For special families of eigenfunctions in negative curvature, we can have:

- If $d = 3$, $\|\psi_h\|_{L^\infty} \geq ch^{-\frac{1}{2}}\|\psi_h\|_{L^2}$.
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For the quantum cat map, there exist families of eigenfunctions saturating the analogue of Bérard's bound. (These families don't satisfy quantum unique ergodicity : Faure-Nonnemacher-de Bièvre '02))

3) New results

L^∞ norms of generic eigenfunctions

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By choosing well the perturbation, we can get $\gamma = \frac{1}{7} - \varepsilon$ if $d = 2$ and $\gamma = \frac{2}{9} - \varepsilon$ if $d = 3$.

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Dream: be able to take for P_h a small perturbation of the metric/ add a small potential/consider a typical metric in the moduli space (if $d = 2$), and get to $\gamma = \frac{d-1}{2} - \varepsilon$.

L^∞ norms of generic eigenfunctions (2)

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We take $P_h^\omega = -h^2\Delta + h^\alpha \mathrm{Op}_h(q_h^\omega)$, where q_h^ω is a symbol which oscillates and decorrelates at scale h^β, with $0 < \alpha, \beta < 1$ well-chosen.

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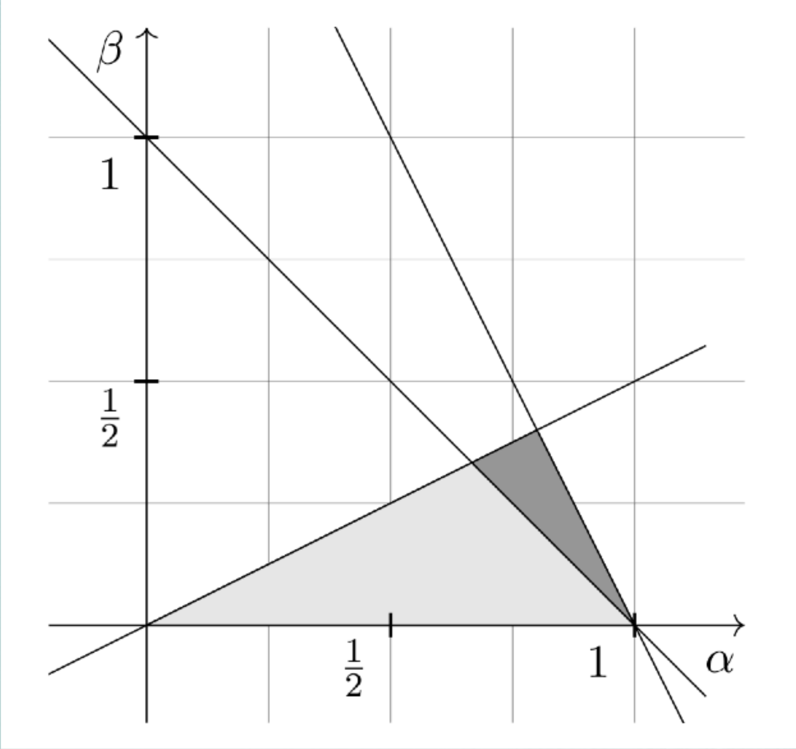

We then set $\widetilde{P}_h^\omega := e^{\frac{it}{h}P_h^\omega}(-h^2\Delta)e^{-\frac{it}{h}P_h^\omega}$ for some $t > 0$.

The result says that, with probability $1 - O(h^\infty)$, whenever $\widetilde{P}_h^\omega\psi_h = \lambda\psi_h$ for some $\lambda \in (\frac{1}{2}, 2)$, we have $\|\psi_h\|_{L^\infty} \leq Ch^{\frac{1-d}{2}+\gamma}\|\psi_h\|_{L^2}$.

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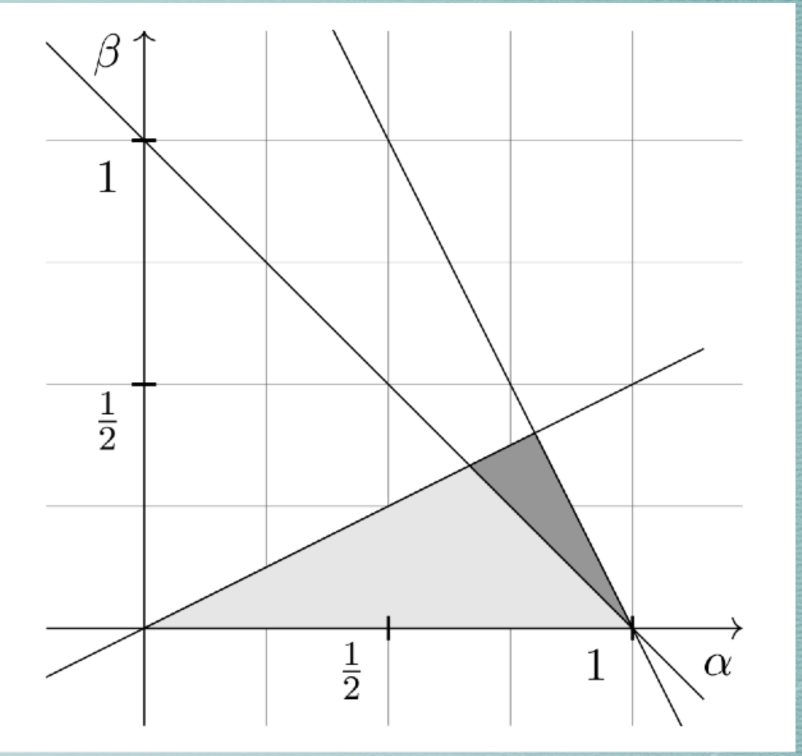

If $\rho = (x,\xi)$, take $q_h^\omega(\rho) = \sum_{j\in J_h} \omega_j \chi(h^{-\beta}\mathrm{dist}(\rho,\rho_j))$, where $\bigcup_{j\in J_h} \mathrm{supp}\left(\chi(h^{-\beta}\mathrm{dist}(\cdot,\rho_j))\right)$ is a locally finite cover of T^*X

We then set $\widetilde{P}_h^\omega := e^{\frac{it}{h}P_h^\omega}(-h^2\Delta)e^{-\frac{it}{h}P_h^\omega}$ for some $t > 0$. By Egorov's theorem, $\widetilde{P}_h^\omega = -h^2\Delta + h^\alpha \mathrm{Op}_h(\widetilde{q}_h^\omega)$, where $\widetilde{q}_h^\omega$ is a symbol which oscillates at scale h^β.

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The eigenfunctions of $\widetilde{P}_h^\omega \widetilde{\psi}_h = \widetilde{\psi}_h$ are of the form $e^{itP_h^\omega}\psi_h$, where $-h^2\Delta\psi_h = \psi_h$.

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Theorem (*I.-Chaubet, work in progress*): (X, g) compact Riemannian manifold of **constant negative curvature**.

$$\left(-h^2\Delta\psi_h = \psi_h\right) \implies \|\psi_h\|_{L^\infty} = o\left(\frac{h^{\frac{1-d}{2}}}{\sqrt{|\log h|}}\|\psi_h\|_{L^2}\right).$$

III. Ideas of proof

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Approach 1: *(Bourgain, Lindenstrauss, Sarnak...)*
Use the fact that you have an explicit expression for the eigenfunctions, or that they are eigenfunctions of an auxiliary operator.
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Approach II: *(Berard, Anantharaman, Nonnenmacher, Dyatlov, Jin...)*

1) Choose a ``basis" of functions $(f_{h,k})_k$ in which to express the eigenfunctions ψ_h. (For instance, Dirac masses, WKB states, or Gaussian coherent states).
2) Understand the evolution of $f_{h,k}$ by some wave equation. For instance, consider the propagator $e^{-i\frac{t}{h}P_h}f_{h,k}$ with $P_h = -h^2\Delta$. Here, one should take t as large as possible.
3) Deduce some nice things about the eigenfunctions ψ_h from information on $e^{-i\frac{t}{h}P_h}$.

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The eigenfunctions $\widetilde{P}_h^\omega \widetilde{\psi}_h = \widetilde{\psi}_h$ are of the form $e^{i\frac{t}{h}P_h^\omega}\psi_h$, where $-h^2\Delta\psi_h = \psi_h$.

Here, $P_h^\omega = -h^2\Delta + h^\alpha \mathrm{Op}_h(q_h^\omega)$, where q_h^ω is a symbol which oscillates and decorrelates at scale h^β, with $0 < \alpha, \beta < 1$ well-chosen.

III. Ideas of proof

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We only take t bounded in step 2.

No step 3!

Here, $P_h^{\omega} = -h^2\Delta + h^{\alpha}\mathrm{Op}_h(q_h^{\omega})$, where q_h^{ω} is a symbol which oscillates and decorrelates at scale h^{β}, with $0 < \alpha, \beta < 1$ well-chosen.

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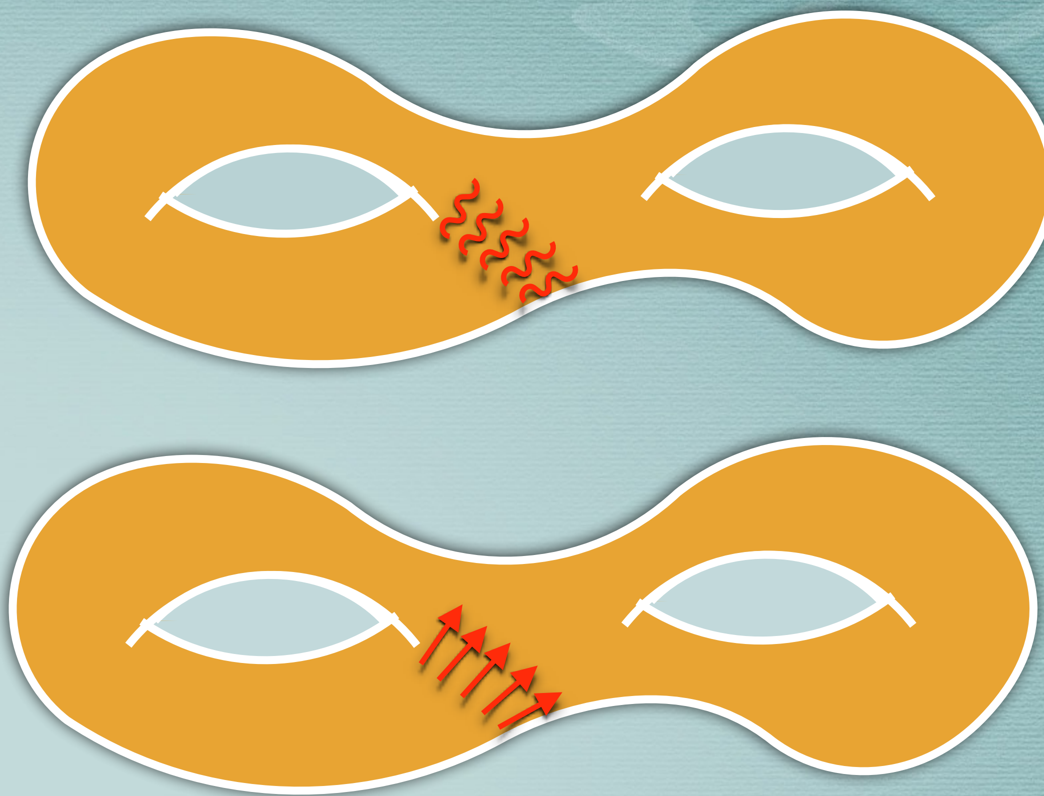
Using Lagrangian states (Step 1.1)

(Monochromatic) Lagrangian states:

$$f_h(x) = a(x)e^{\frac{i}{h}\varphi(x)}, \quad \text{with } |\nabla\varphi| \equiv c, \quad a \in C_c^\infty(X) \qquad (1)$$



f_h is microlocalised on

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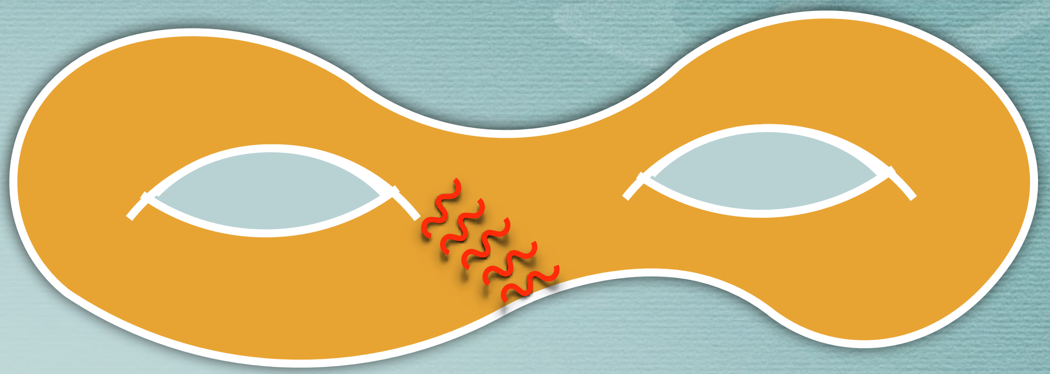

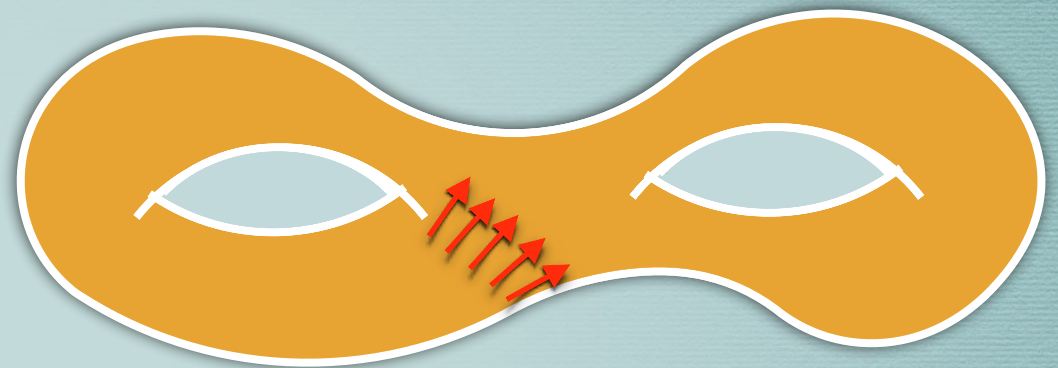

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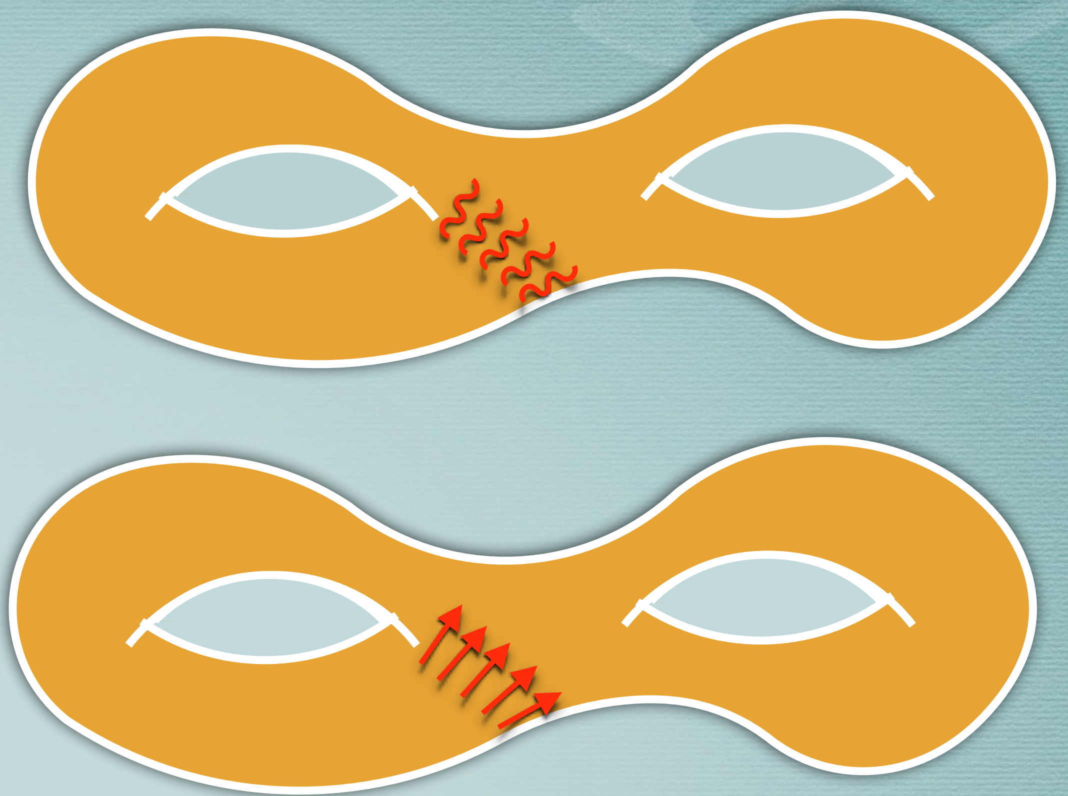

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$$\chi\psi = \sum_{\substack{n\in\mathbb{Z}^d \\ |n|^2\approx\lambda^2}} a_n e^{inx}$$

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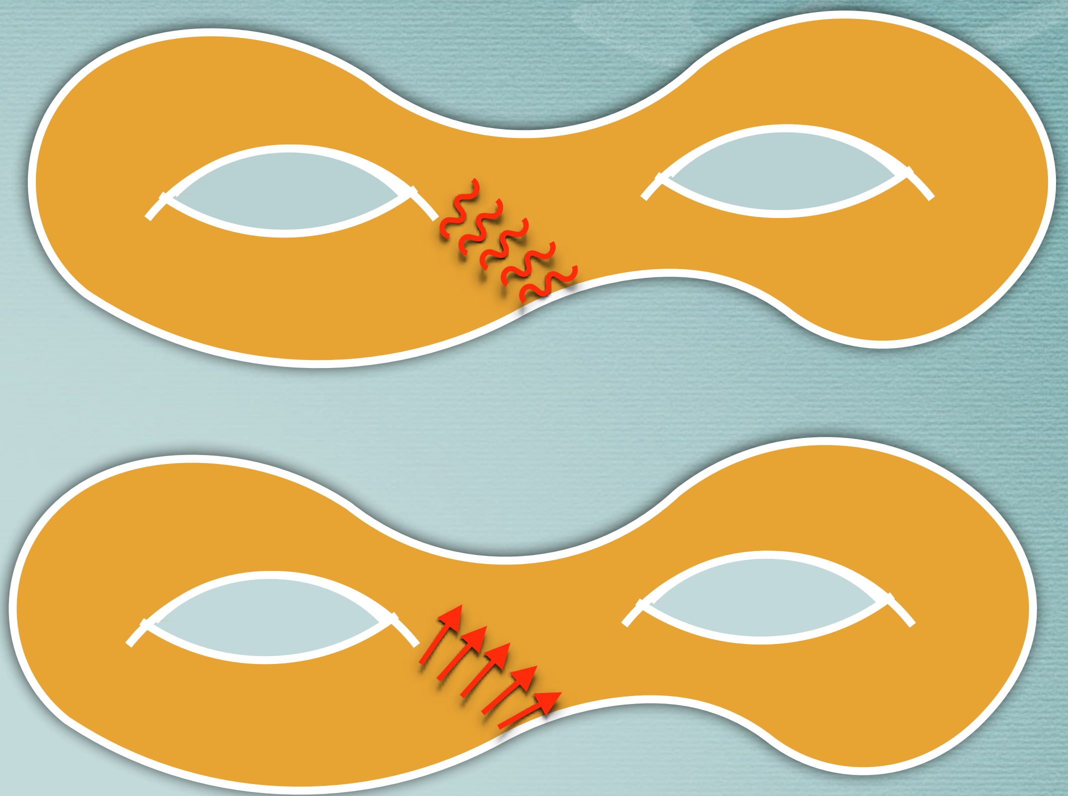

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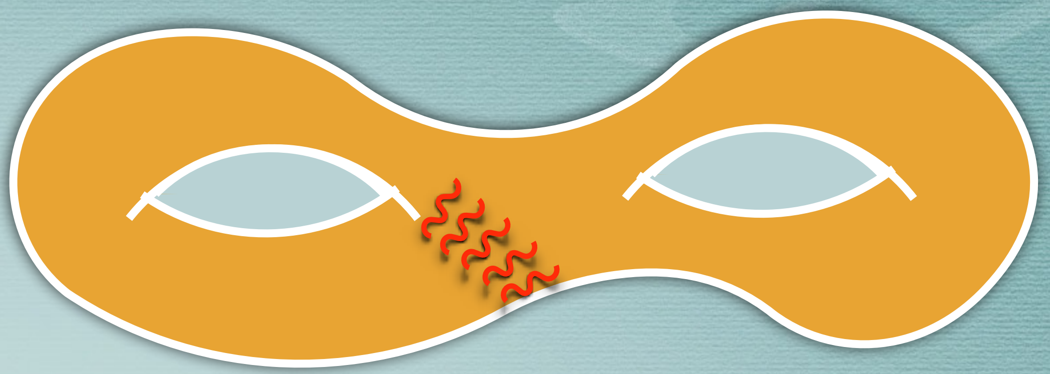

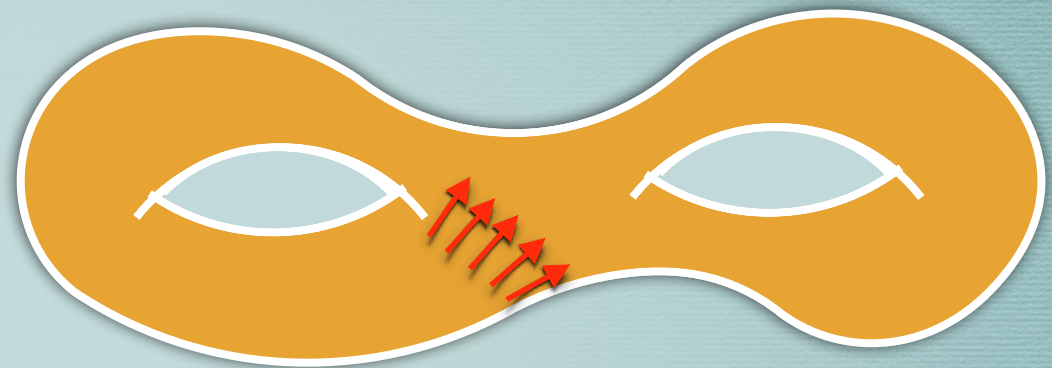

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We see that $|\psi_h(x)|$ is large if:

- Most b_k are not too small.
- And the $b_k a_k(x)e^{\frac{i}{h}\varphi_k(x)}$ have similar phases.

Propagating Lagrangian states (Step 2)

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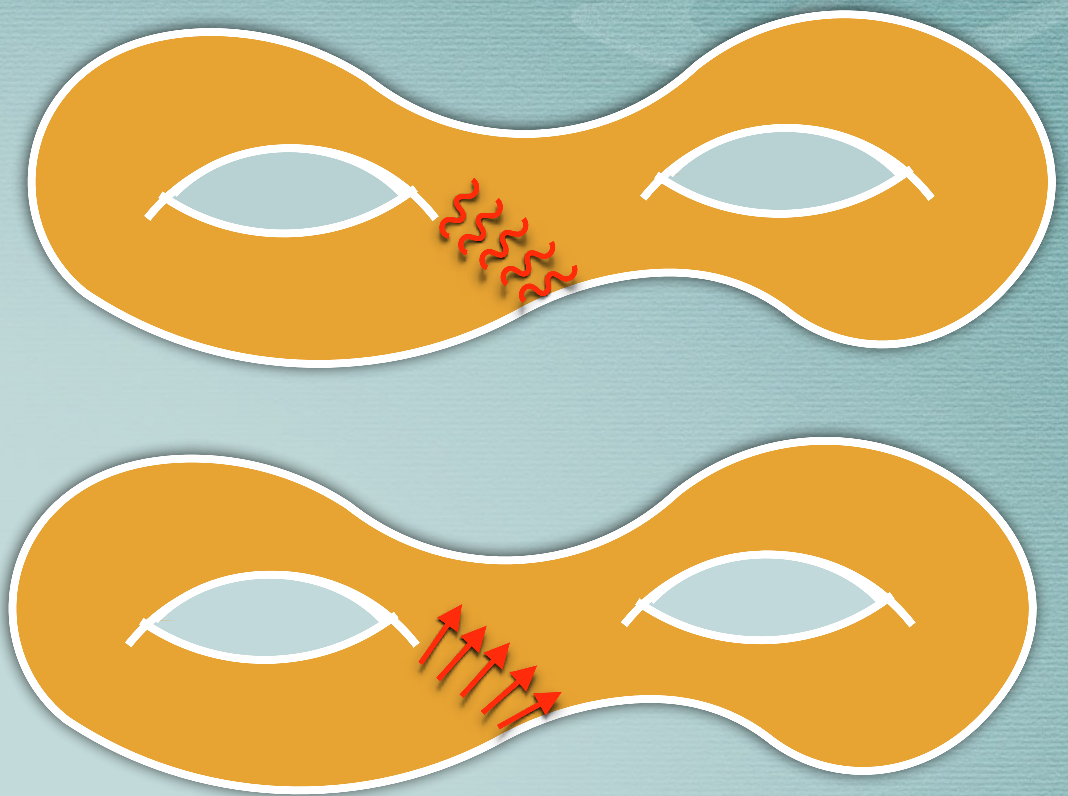

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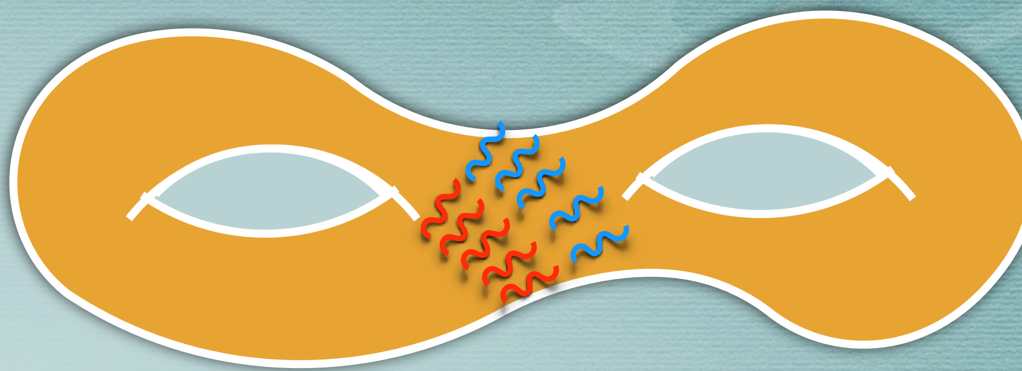

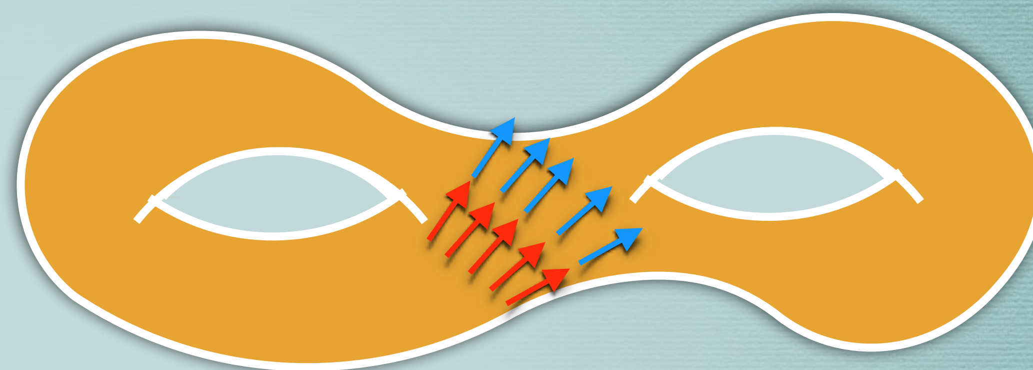

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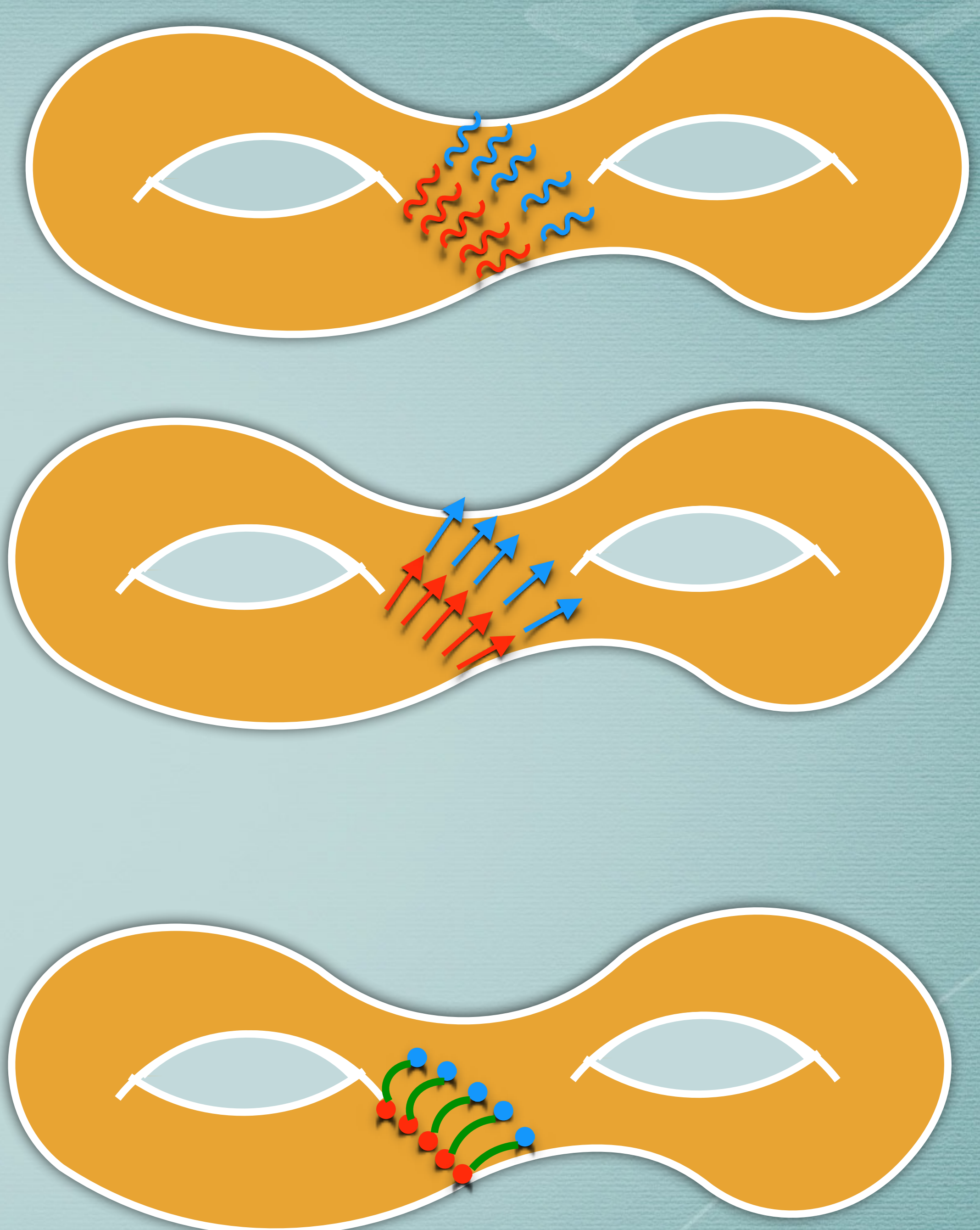

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- $$\theta_h^\omega(x,t) = h^{\alpha-1}\int_0^t q_\omega \circ \Phi^s \mathrm{d}s$$

Regrouping Lagrangian states (Step 1.2)

$-h^2\Delta\psi_h = \psi_h$, decomposed as $\psi_h = \sum_{k\in K_h} b_k f_{k,h}$.

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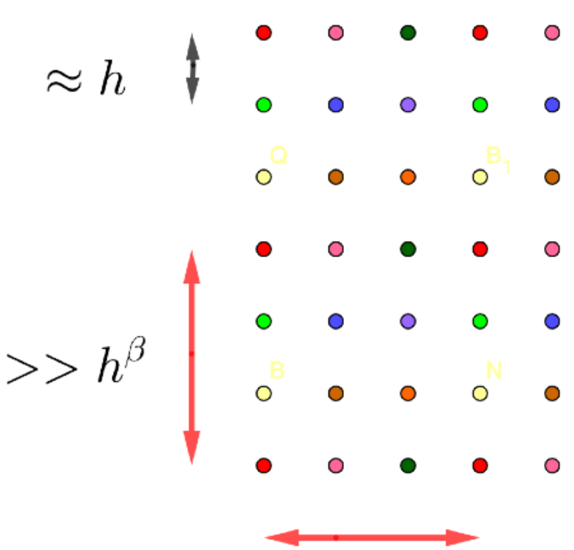

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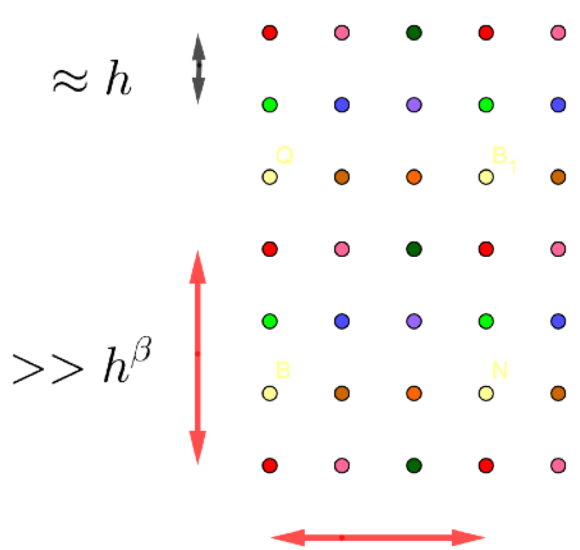

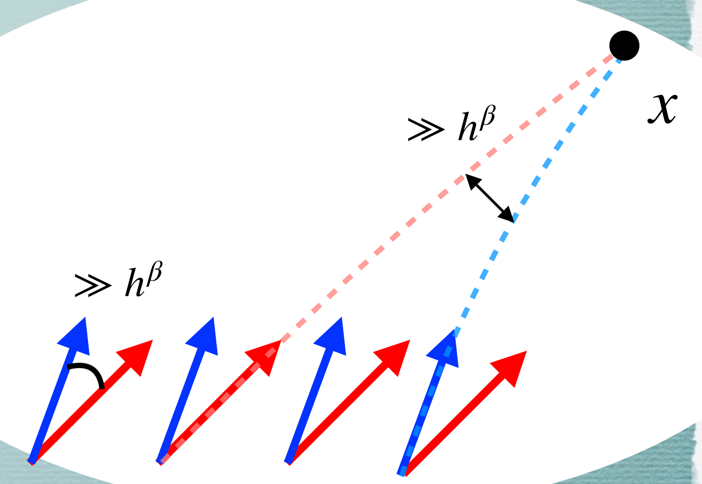

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With probability $1 - O(h^\infty)$,

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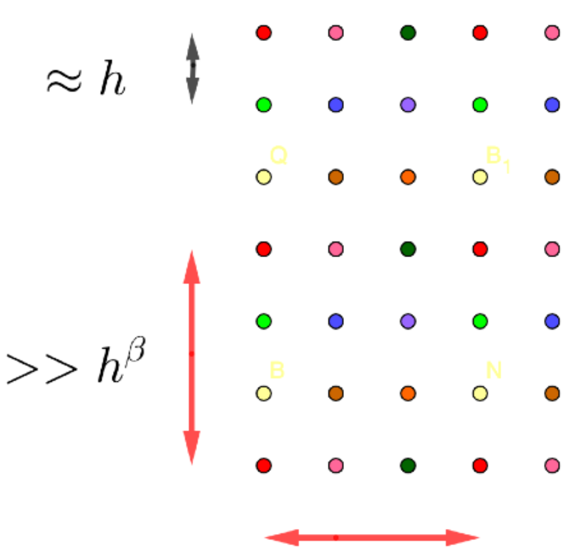

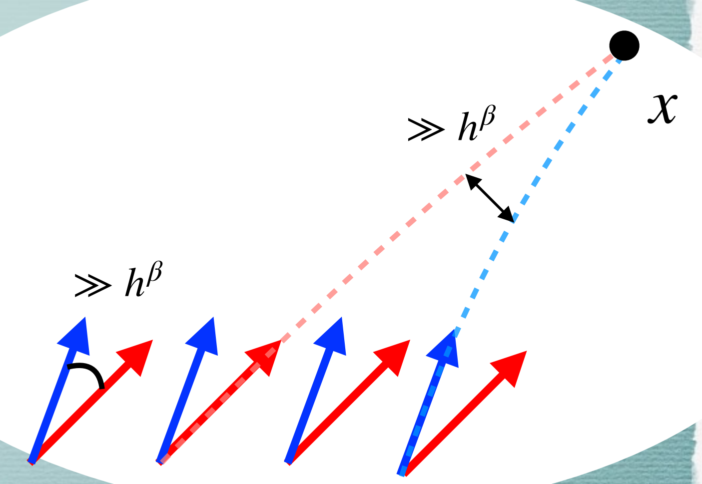

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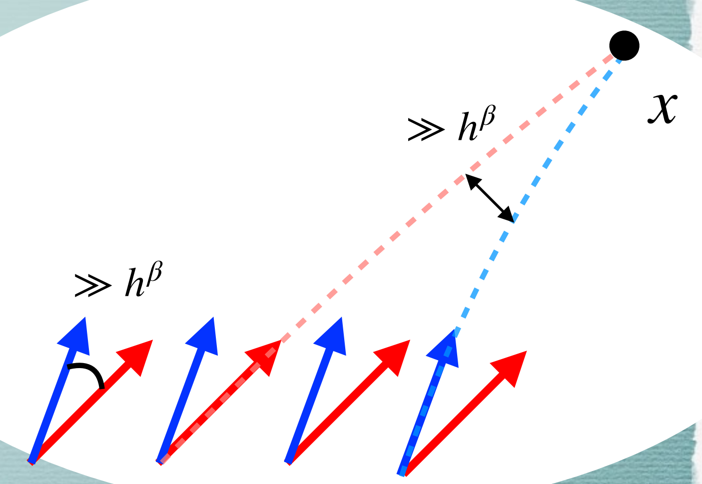

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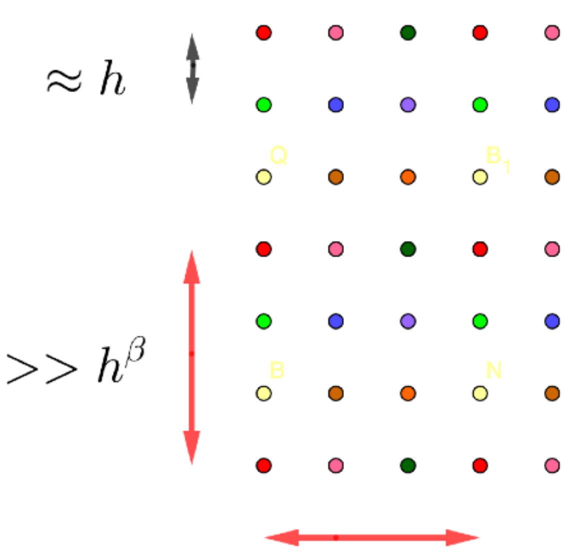

With probability $1 - O(h^{\infty})$, $|\psi_h(x)| \leq h^{-\varepsilon}\sqrt{|I_h|}$.

III. Ideas of proof

2) Bounds on deterministic eigenfunctions

1) Choose a ``basis" of functions $(f_{h,k})_k$ in which to express the eigenfunctions ψ_h. (For instance, Dirac masses, WKB states, or Gaussian coherent states).
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For the quantum cat map, $t \mapsto e^{-i\frac{t}{h}P_h}$ is $|\log h|$-periodic

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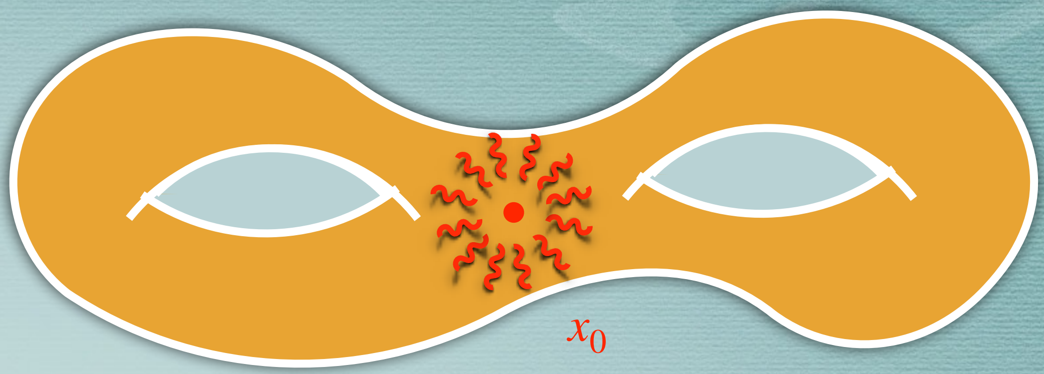
From echo estimates to L^∞ estimates

Step 1

Radial Lagrangian state:

$f_h(x) = \chi(\mathrm{d}(x, x_0))e^{\frac{i}{h}\mathrm{d}(x,x_0)}$.



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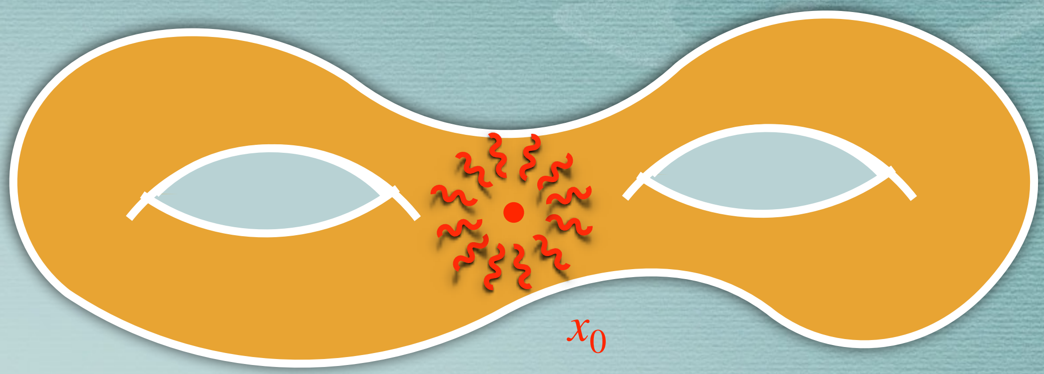

Step 2

Echo estimate: (X, g) compact Riemannian manifold of **constant negative curvature**.
There exists $\gamma > 0$ such that for all $M > 0$, $\exists C_M > 0$ such that for all $t \leq M|\log h|$
$$\|e^{ith\Delta} f_h\|_{L^\infty} \leq C_M h^{\frac{1-d}{2}+\gamma}.$$

Using only Berard's bounds on eigenfunctions, we would only get
$$\|e^{ith\Delta} f_h\|_{L^\infty} \leq \frac{C}{\sqrt{|\log h|}} h^{\frac{1-d}{2}}.$$

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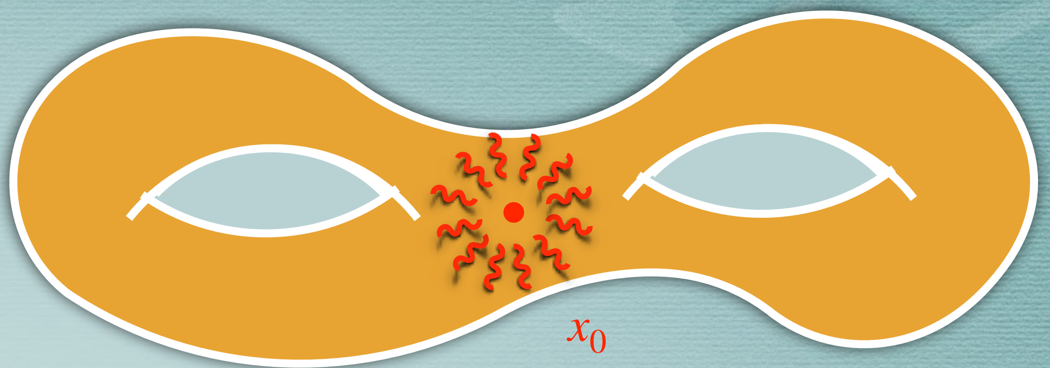

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Proof of the L^∞ estimate on eigenfunctions

Suppose $-h^2\Delta\psi_h = \psi_h$.

- Fact : $|\psi_h(x_0)|^2 = Ch^{1-d}|\langle \psi_h, f_h\rangle|^2$.
- Echo estimate $\Longrightarrow$ The family $f_{n,h} := e^{ihn\Delta} f_h$, $n \leq M|\log h|$ is (almost) orthogonal.
- $|\langle f_h, \psi_h\rangle|^2 = |\langle f_{n,h}, \psi_h\rangle|^2 \;\; \forall n \in \mathbb{N}$.
- Parseval's formula.

Step 3

Proof of the echo estimate

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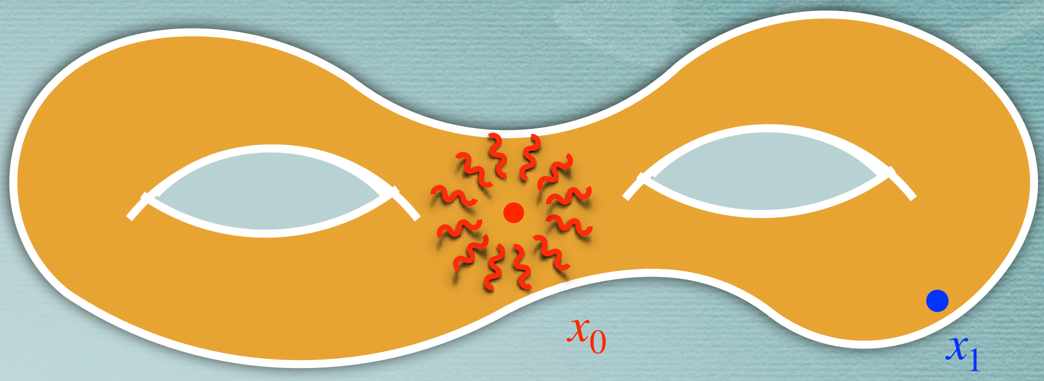
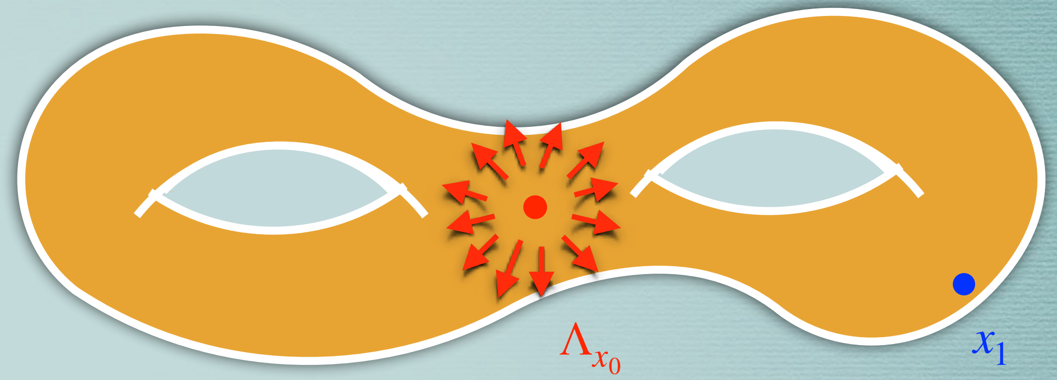
Microlocalised on a submanifold:

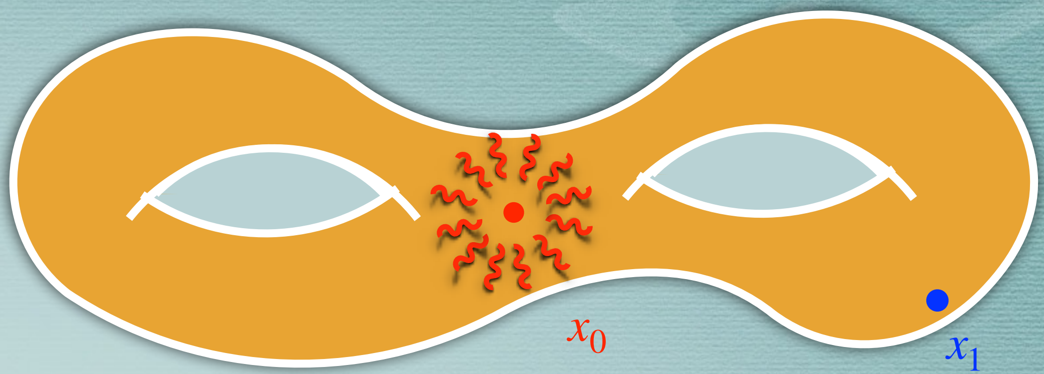
$$\Lambda_{x_0} := \bigcup_{t\in I} \Phi^t(S^*_{x_0}).$$



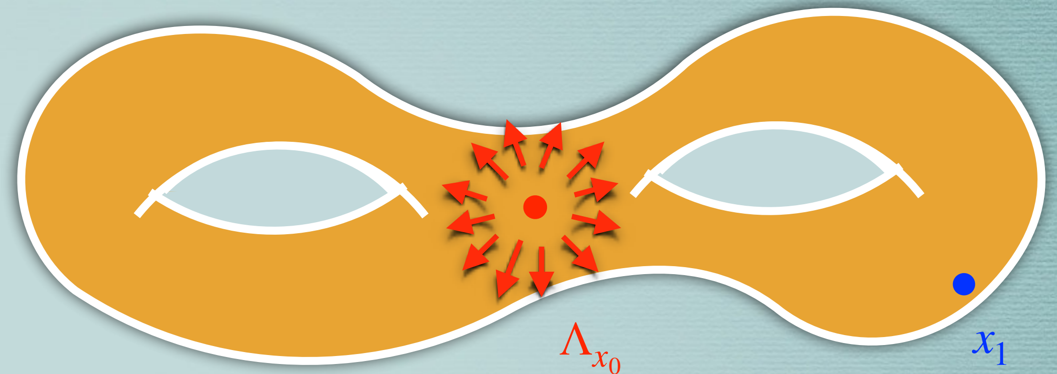


Proof of the echo estimate

Radial Lagrangian state:
$f_h(x) = \chi(\mathrm{d}(x, x_0))e^{\frac{i}{h}\mathrm{d}(x,x_0)}$.



Microlocalised on a submanifold:
$$\Lambda_{x_0} := \bigcup_{t \in I} \Phi^t(S^*_{x_0}).$$



For small t, we have $(e^{ith\Delta} f_h)(x_1) \approx a_h(t, x_1)e^{\frac{i}{h}\varphi_t(x_1)}$, with $\{(x_1, \nabla \varphi_t(x_1))\} = \Phi^t(\Lambda_{x_0})$, where Φ^t is the geodesic flow, and $a_h(t, x_1)$ satisfies a transport equation.

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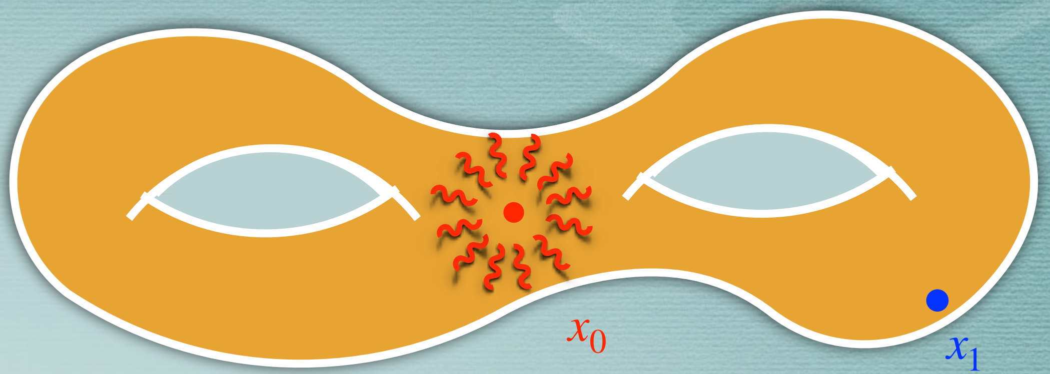

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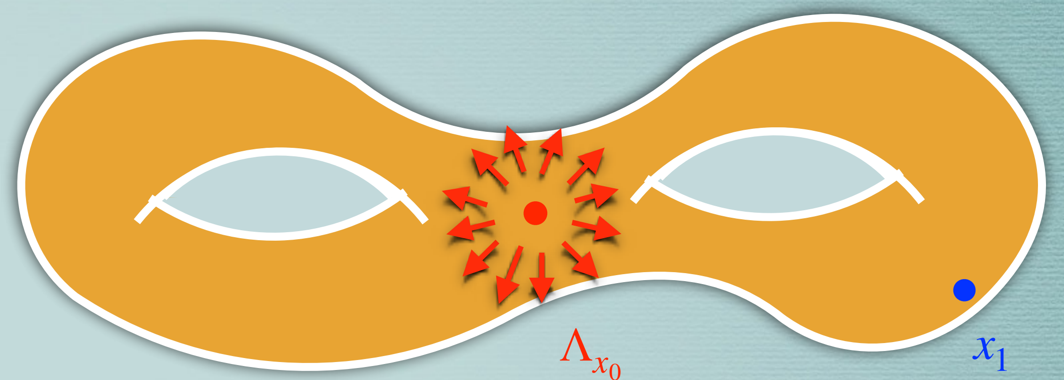
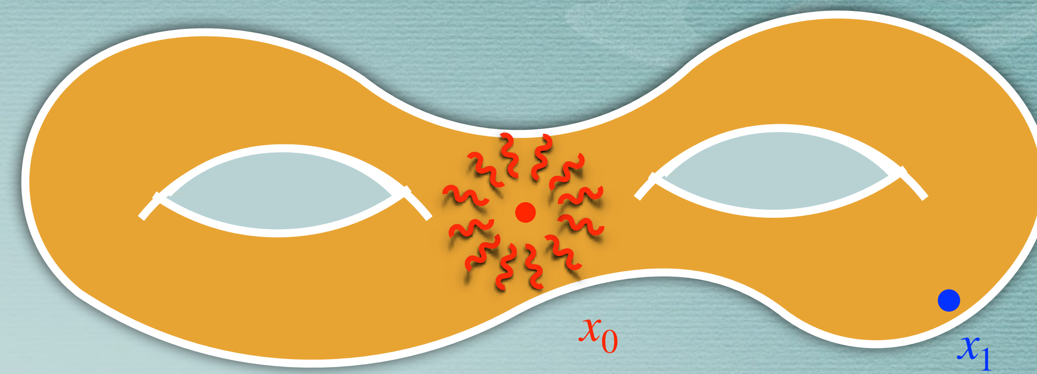


For small t, we have $(e^{ith\Delta}f_h)(x_1) \approx a_h(t, x_1)e^{\frac{i}{h}\varphi_t(x_1)}$, with $\{(x_1, \nabla\varphi_t(x_1))\} = \Phi^t(\Lambda_{x_0})$, where Φ^t is the geodesic flow, and $a_h(t, x_1)$ satisfies a transport equation.

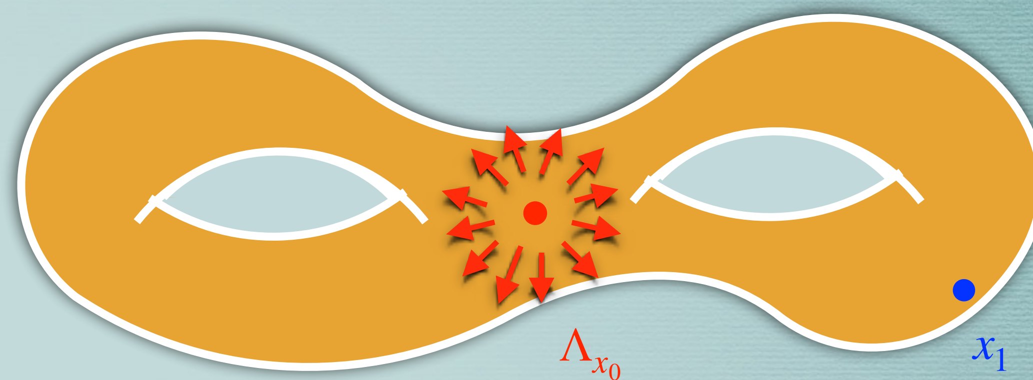
For larger t, we have $(e^{ith\Delta}f_h)(x_1) \approx \sum_j a_{j,h}(t, x_1)e^{\frac{i}{h}\varphi_{j,t}(x_1)}$, where the sum contains $O(e^{(d-1)t})$ terms, and each $a_{j,h}(t, \cdot\,)$ is of size $O(e^{(1-d)t/2})$.

Proof of the echo estimate

Radial Lagrangian state:
$f_h(x) = \chi(\mathrm{d}(x, x_0))e^{\frac{i}{h}\mathrm{d}(x,x_0)}$.



Microlocalised on a submanifold:
$$\Lambda_{x_0} := \bigcup_{t\in I} \Phi^t(S^*_{x_0}).$$



For small t, we have $(e^{ith\Delta}f_h)(x_1) \approx a_h(t, x_1)e^{\frac{i}{h}\varphi_t(x_1)}$, with $\{(x_1, \nabla\varphi_t(x_1))\} = \Phi^t(\Lambda_{x_0})$, where Φ^t is the geodesic flow, and $a_h(t, x_1)$ satisfies a transport equation.

For larger t, we have $(e^{ith\Delta}f_h)(x_1) \approx \sum_j a_{j,h}(t, x_1)e^{\frac{i}{h}\varphi_{j,t}(x_1)}$, where the sum contains $O(e^{(d-1)t})$ terms, and each $a_{j,h}(t, \cdot\,)$ is of size $O(e^{(1-d)t/2})$.

For t large enough, $|\,(e^{ith\Delta}f_h)(x_1)\,| = e^{\frac{t}{2}(d-1)}\left|\left\langle \phi^t_*\left(e^{\frac{i}{h}\mathrm{d}(\cdot,x_0)}G_{t,x_0}\right), \mu_{x_1}\right\rangle_{L^2}\right| + O(h^\infty)$, where

- ϕ^t is the classical flow on $M := S^*X$ (and $\phi^t_* f = f \circ \phi^t$).
- G_{t,x_0} is a smooth function, bounded along with all its derivatives, independently of t (it is a regularization of the uniform measure on $\Lambda_{x_0} \subset M$, in the stable directions).
- μ_{x_1} is the uniform measure on $S^*_{x_1}X \subset M$.

Proof of the echo estimate (2)

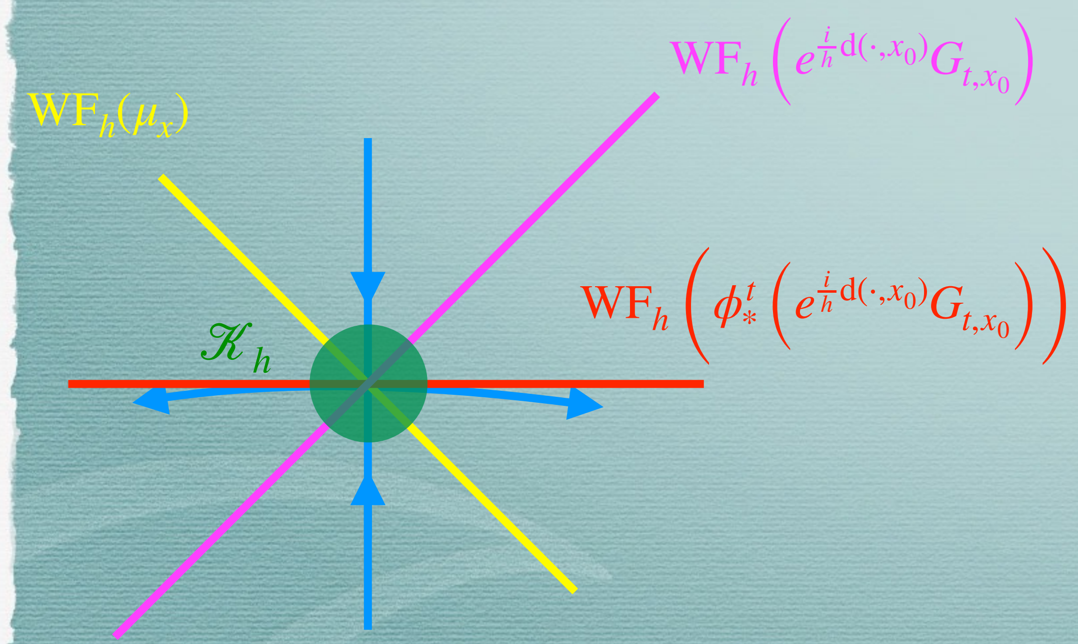
$$|(e^{ith\Delta}f_h)(x_1)| = e^{\frac{t}{2}(d-1)}\left|\left\langle \phi_*^t\left(e^{\frac{i}{h}\mathrm{d}(\cdot,x_0)}G_{t,x_0}\right), \mu_{x_1}\right\rangle_{L^2}\right| + O(h^\infty), \text{ where}$$

- ϕ^t is the classical flow on $M := S^*X$ (and $\phi_*^t f = f \circ \phi^t$).
- G_{t,x_0} is a smooth function, bounded along with all its derivatives, independently of t.
- μ_{x_1} is the uniform measure on $S^*_{x_1}X \subset M$

Idea (Faure-Sjöstrand ≈10): see $\phi_*^t : L^2(M) \longrightarrow L^2(M)$ as a quantum propagator, and thus, as a Fourier Integral Operator over T^*M. The associated classical dynamics is the symplectic lift $\widetilde{\phi}^t : T^*M \longrightarrow T^*M$.

The classical dynamics has a trapped set $K = \{(z,\zeta) \in T^*M; \zeta = \lambda\alpha(z), \lambda > 0\}$, where α is the contact one-form generating the classical dynamics.
In the sequel, we will only consider the subset $K_1 = \{(z,\zeta) \in T^*M; \zeta = \alpha(z)\}$.

All the relevant dynamics happens only in a neighborhood $\mathscr{K}_h$ of size $h^{\frac{1}{2}-\varepsilon}$ of K_1.



If Π_h is a pseudodifferential operator microlocalised in $\mathscr{K}_h$:

- $\|\Pi_h\phi_*^t\Pi_h\|_{L^2\to L^2} \leq Ce^{\left(\frac{(1-d)}{2}+\varepsilon\right)t}$ (Faure-Tsujii, Nonnenmacher-Zworski)
- $\|\Pi_h\mu_{x_1}\|_{L^2} = O(h^{-\frac{d}{4}-c\varepsilon})$
- $\left\|\Pi_h\left(e^{\frac{i}{h}\mathrm{d}(\cdot,x_0)}G_{t,x_0}\right)\right\|_{L^2} = O(1)$
- «Invariance» by the flow gives an extra $O(h^{1/4})$.

Ongoing and future projects

- With A.Garcia Ruiz: Adapt the generic result to the case of a confining potential in $\mathbb{R}^d$ (with a small random pseudodifferential perturbation).
- With M. Vogel: Show more properties of eigenfunctions under generic perturbations (Quantum Unique Ergodicity? Berry's conjecture?)
- With Théophile Chaumont-Frelet: Perform numerical experiments for $\|\psi_h\|_{L^\infty}$ in variable curvature.

Thank you for your attention!