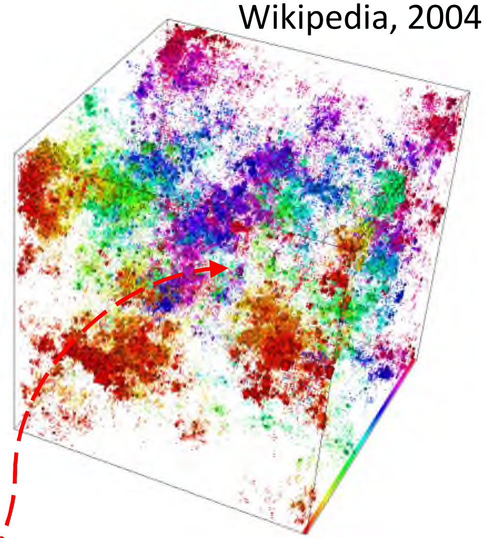
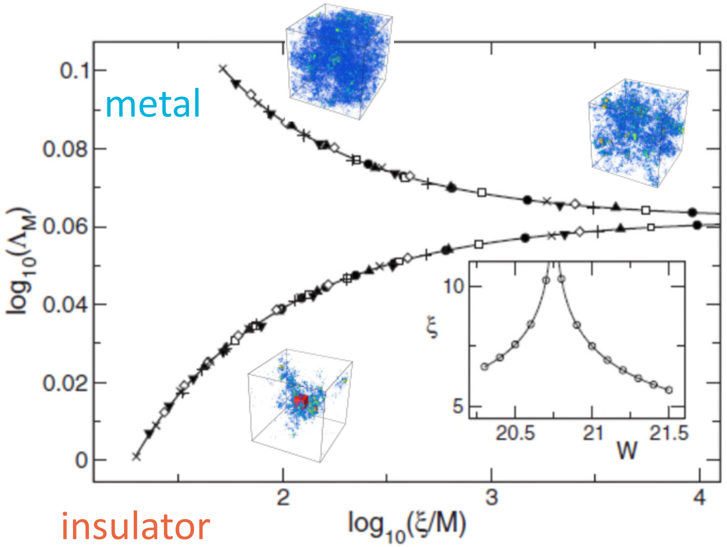
The 3D Anderson model with disorder

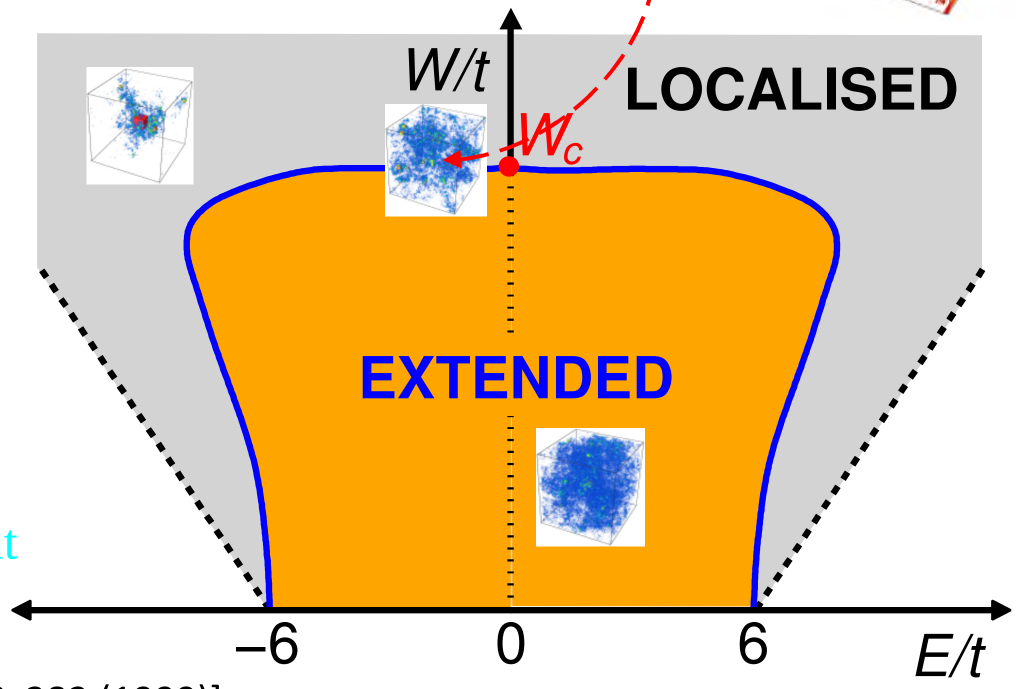
Wikipedia, 2004



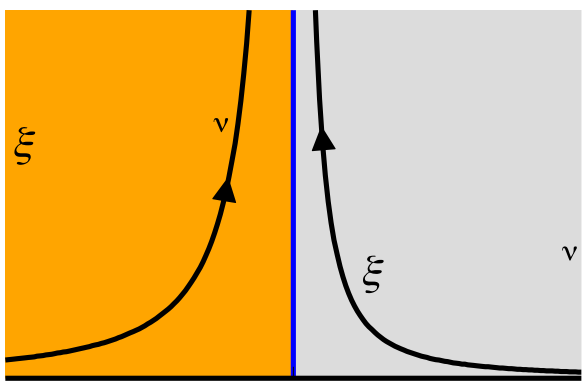
Kramer, B., & MacKinnon, A. (1993). Localization: theory and experiment. *Reports on Progress in Physics*, *56*(12), 1469–1564.



- Phase diagram in 3D



- Divergent localization length



$$\xi \sim |X - X_c|^{-\upsilon}$$

with $X = E$ or W

υ = critical exponent

$\nu = 1.590(579{,}602)$

[Slevin+Ohtsuki, PRL *82*, 382 (1999)]

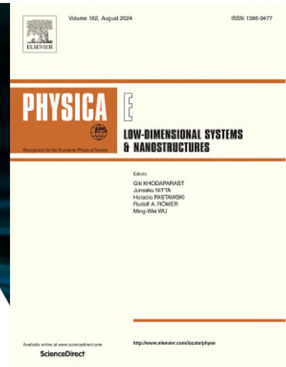
Spectral measures for Anderson localization in variants of the “standard model”

Flat bands, quasi-periodic electron models and many-body interactions

RA Römer, and many others (see later citations …)

$$H = \sum_{\boldsymbol{r}} \varepsilon_{\boldsymbol{r}} |\boldsymbol{r}\rangle\langle\boldsymbol{r}| - \sum_{\langle \boldsymbol{r} \neq \boldsymbol{r}' \rangle} t_{\boldsymbol{r},\boldsymbol{r}'} |\boldsymbol{r}\rangle\langle\boldsymbol{r}'|$$

STROM, Nov 2025



Works published in

[warwick.ac.uk/rudoroemer/publications]

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Flat bands:

[203, 197] + [194] "Unconventional delocalization in a family of 3D Lieb lattices“, J. Liu, C. Danieli, J. Zhong, RAR, Phys. Rev. B **106**, 214204 (2022)+ [186,182,180]

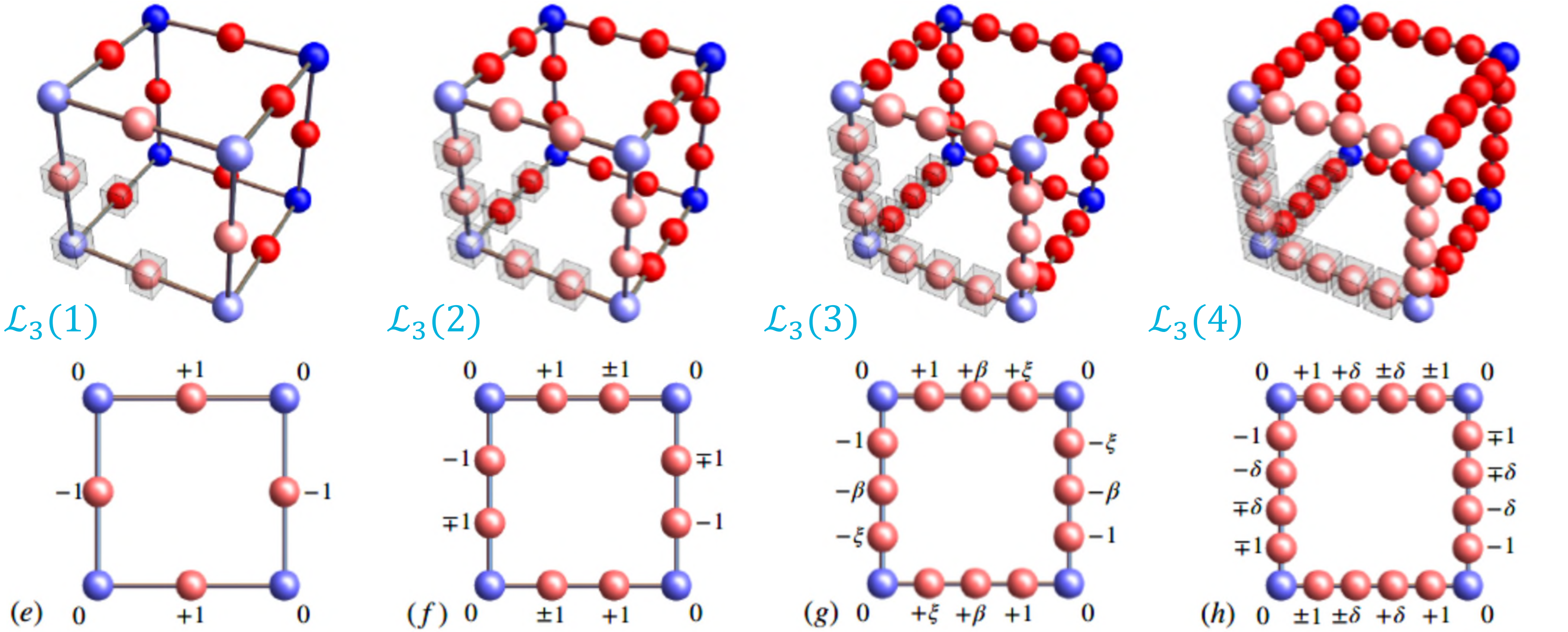
Quasi-periodic AB tiling:

[189] "The GOE ensemble for quasiperiodic tilings without unfolding: r-value statistics“, U. Grimm, RAR, Phys. Rev. B 104(6), L060201 (2021); [20] "Level Spacings Distributions of Planar Quasiperiodic Tight-Binding Models“, J. X. Zhong, U. Grimm, RAR, M. Schreiber, Phys. Rev. Lett. 80, 3996-3999 (1998).

Many-body and disorder:

[200] "Spectral and Entanglement Properties of the Random Exchange Heisenberg Chain“, Y. Gao, RAR, Phys. Rev. B 111, 104202 (2025)

Compactly localized states (CLS) imply flat bands (Ex: Lieb models)



$E_{\mathrm{CLS}} = 0$

$E_{\mathrm{CLS}} = \pm 1$

$E_{\mathrm{CLS}} = \beta = 0, \pm\sqrt{2},$

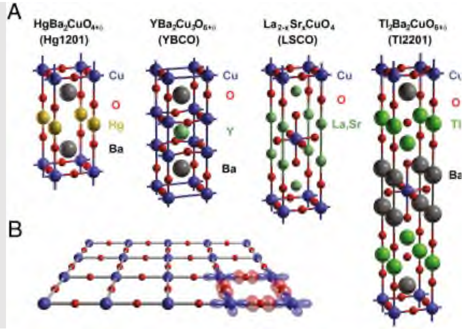
$\xi = 1$, for $\beta = \pm\sqrt{2},$
$\xi = -1$, for $\beta = 0,$

$E_{\mathrm{CLS}} = \pm\delta,$

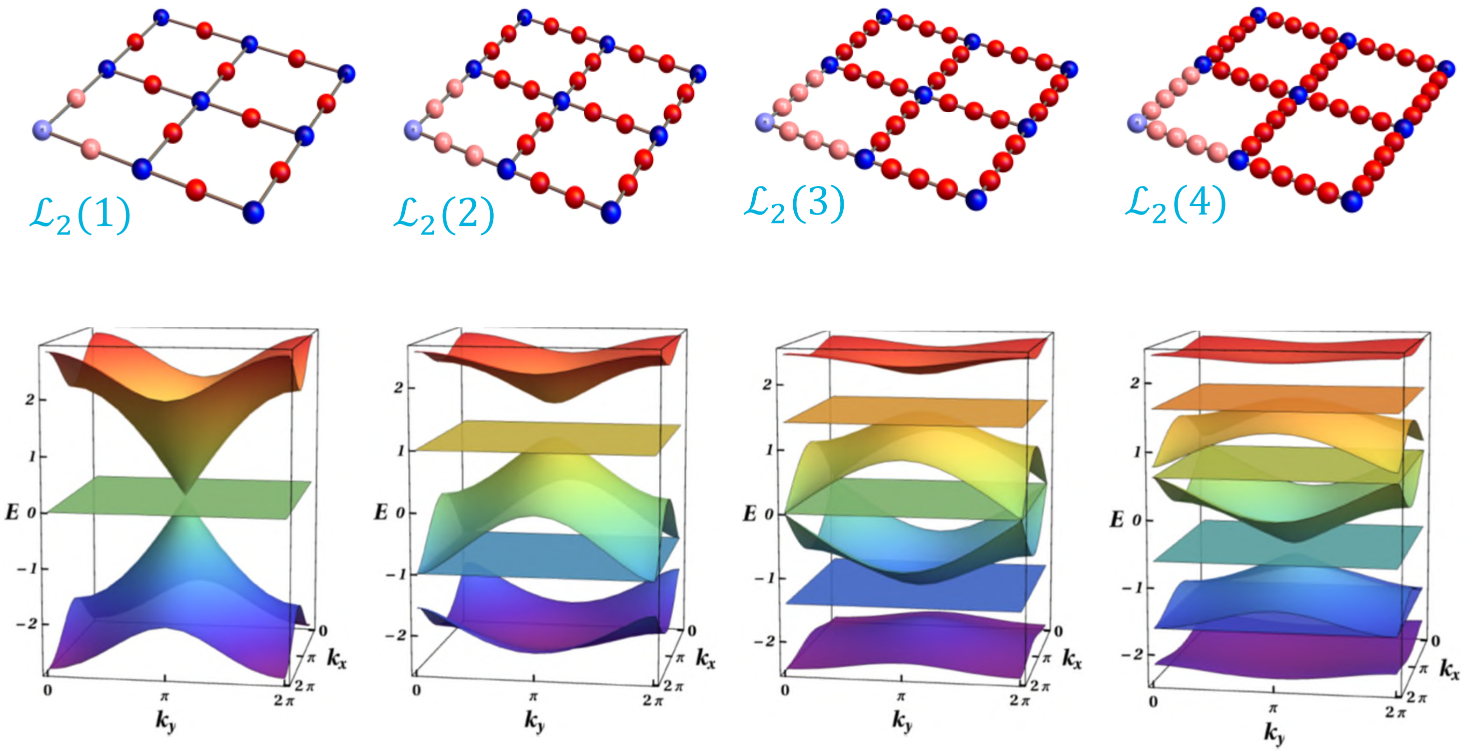
$\delta = (1 \pm \sqrt{5})/2$

High-Tc SCs

CuOx planes



[Lieb model in 2D](#) and its extensions, **the clean case**



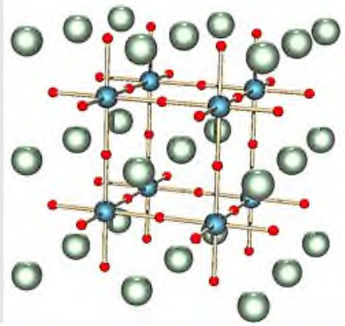
[also Da Zhang, Yiqi Zhang, Hua Zhong, Changbiao Li, Zhaoyang Zhang, Yanpeng Zhang, Milivoj R. Belić, "New edge-centered photonic square lattices with flat bands", Annals of Physics **382** (2017), 160-169]

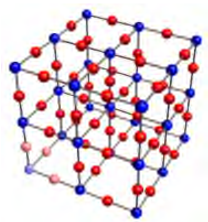
- $\mathcal{L}_2(n)$ exhibits
  - n flat bands and
  - $n+1$ dispersive bands
- Simple **"square lattice"** structure makes it straight-forward to study
- Ideal test case for **flat band physics**

Lieb model in 3D and its extensions, **the clean case**

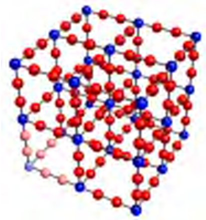
ABX3 Perovskite

X = Lieb sites
B = cube sites
A = not present

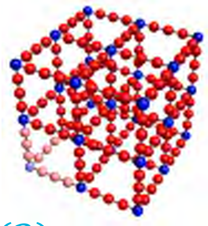




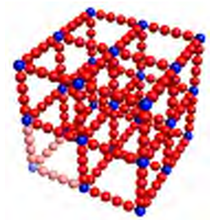
$\mathcal{L}_3(1)$



$\mathcal{L}_3(2)$

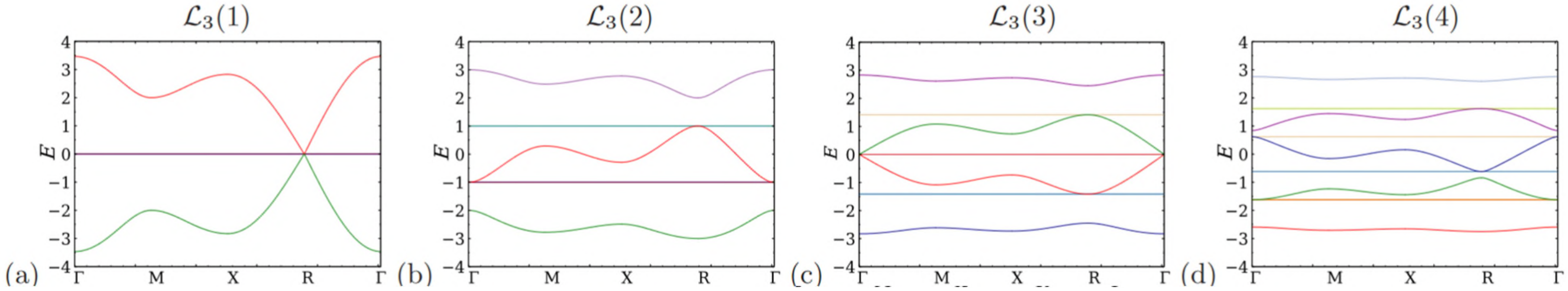


$\mathcal{L}_3(3)$



$\mathcal{L}_3(4)$

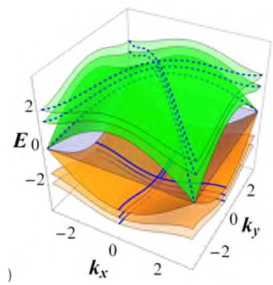
- $\mathcal{L}_3(n)$ exhibits
  - n flat bands and
  - $n+1$ dispersive bands



(0,0,0) $(\pi,\pi,0)$ $(0,\pi,0)$ (π,π,π)

Simple **“square lattice”** structure makes it straight-forward to study

$$H = \sum_{\boldsymbol{r}} \varepsilon_{\boldsymbol{r}} |\boldsymbol{r}\rangle\langle\boldsymbol{r}| - \sum_{\langle \boldsymbol{r}\neq\boldsymbol{r}'\rangle} t_{\boldsymbol{r},\boldsymbol{r}'} |\boldsymbol{r}\rangle\langle\boldsymbol{r}'|$$

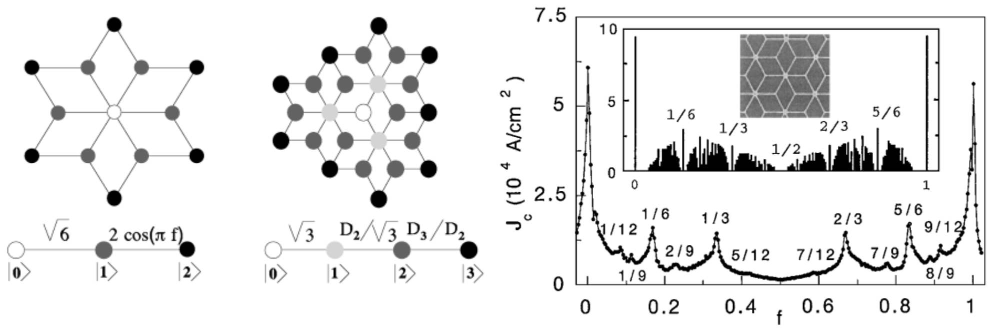


- Ideal test case for **flat band physics in 3D**

Further experimental realizations

Electronic flat band

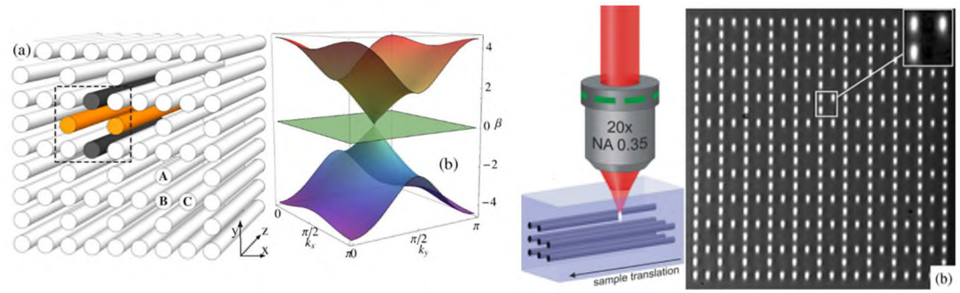
[1] J. Vidal et al., Phys. Rev. Lett. 81, 5888 (1998).
[2] C. C. Abilio et al., Phys. Rev. Lett. 83, 5102 (1999).



The first experiment： Superconducting wire network

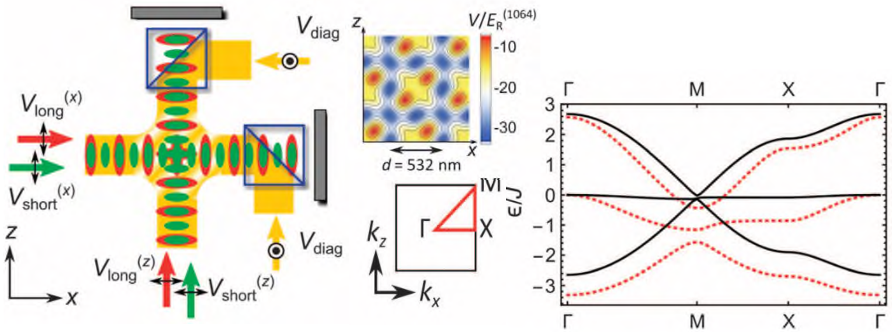
Photonic flat band

[1] Guzmán-Silva, New J. Phys. **16**, 063061 (2014).
[2] R. A. Vicencio et al., Phys. Rev. Lett. 114, 245503 (2015).
[3] S. Mukherjee et al., Phys. Rev. Lett. 114, 245504 (2015).



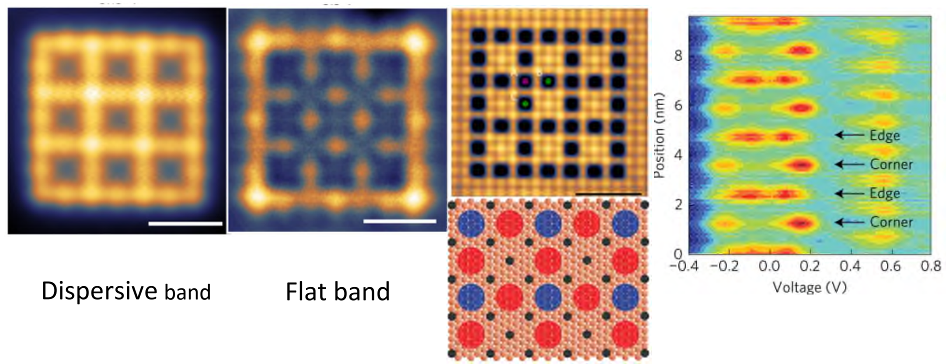
Ultra-cold atoms in an optical flat band

[1] R. Shen, Phys. Rev. B 81, 041410 (2010).
[2] V. Apaja, Phys. Rev. A 82, 041402 (2010).
[3] S. Taie er al., Sci. Adv. 1, e1500854 (2015).



Atomic flat band

[1] R. Drost et al., Nat. Phys. 13, 668 (2017).
[2] M. R. Slot et al., Nat. Phys. 13, 672 (2017).



Dispersive band   Flat band

STM for chlorine monolayer on a Cu(100) (surface)

The fate of the compactly-localize states (CLSs)

- What happens in the presence of disorder?
  - 2D: [180] "Disorder effects in the two-dimensional Lieb lattice and its extensions“, X. Mao, J. Liu, J. Zhong, RAR, Physica E 124, 114340 (2020)
  - 3D: [182] "Localization, phases and transitions in the three-dimensional extended Lieb lattices“, J. Liu, X. Mao, J. Zhong, RAR, Phys. Rev. B 102, 174207 (2020)

- What happens for CLS-preserving disorder?
  - [194] "Unconventional delocalization in a family of 3D Lieb lattices“, J. Liu, C. Danieli, J. Zhong, RAR Phys. Rev. B **106**, 214204 (2022)

- Can we engineer CLS-preserving “Lieb meta-materials”?
  - [197]"Quantum engineering for compactly localized states in disordered Lieb lattices“, C. Danieli, J. Liu, RAR, Eur. Phys. J. B 97, 128 (2024), arXiv:2309.04227
- How to load, store and read-out quantum states via CLSs?
  - Current work: C. Danieli, J. Liu, RAR, **R. A. Vicencio,** https://doi.org/10.48550/arXiv.2508.01846

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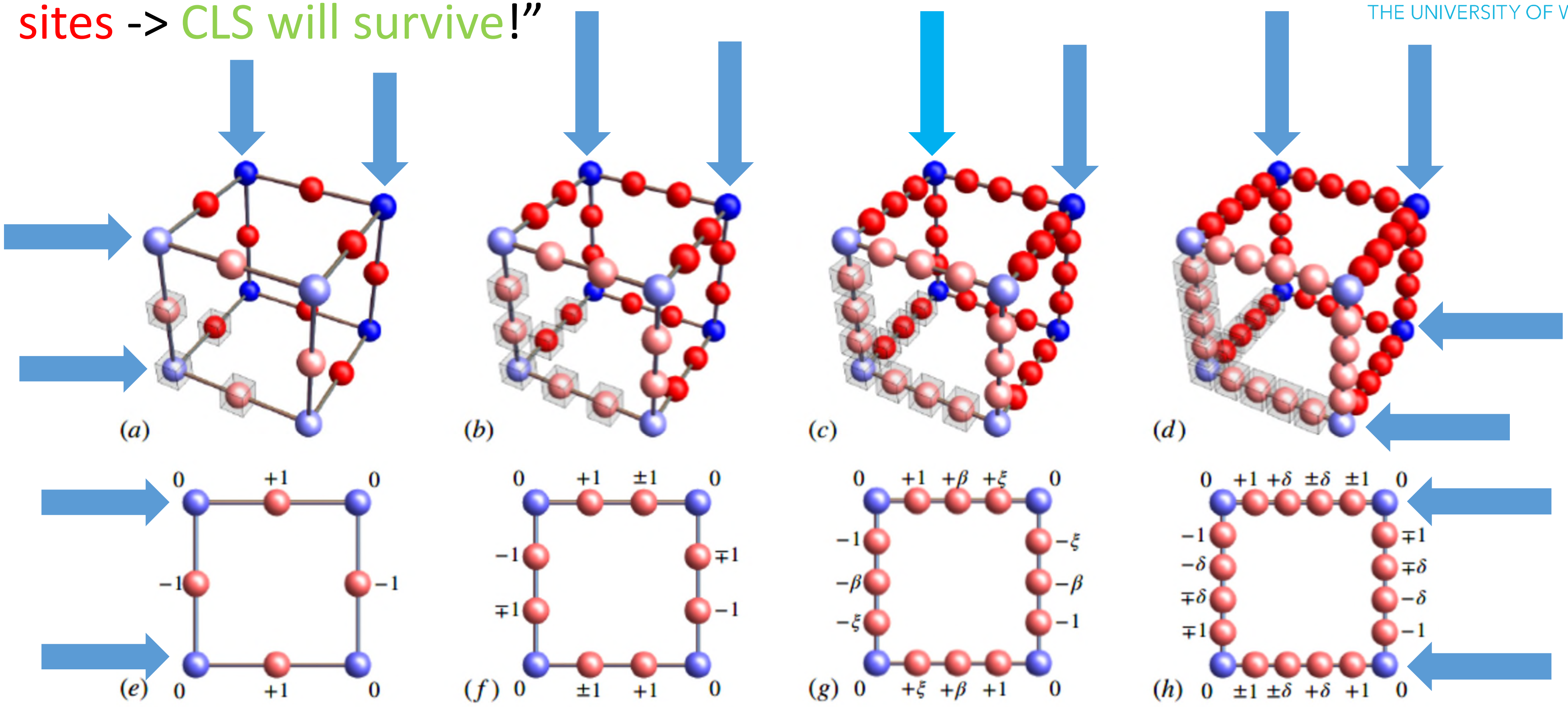
CLS-preserving disorder?

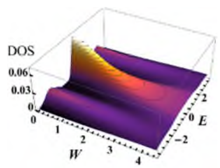
- special disorder at cube sites only, no disorder at Lieb sites -> CLS will survive!”

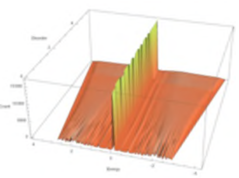


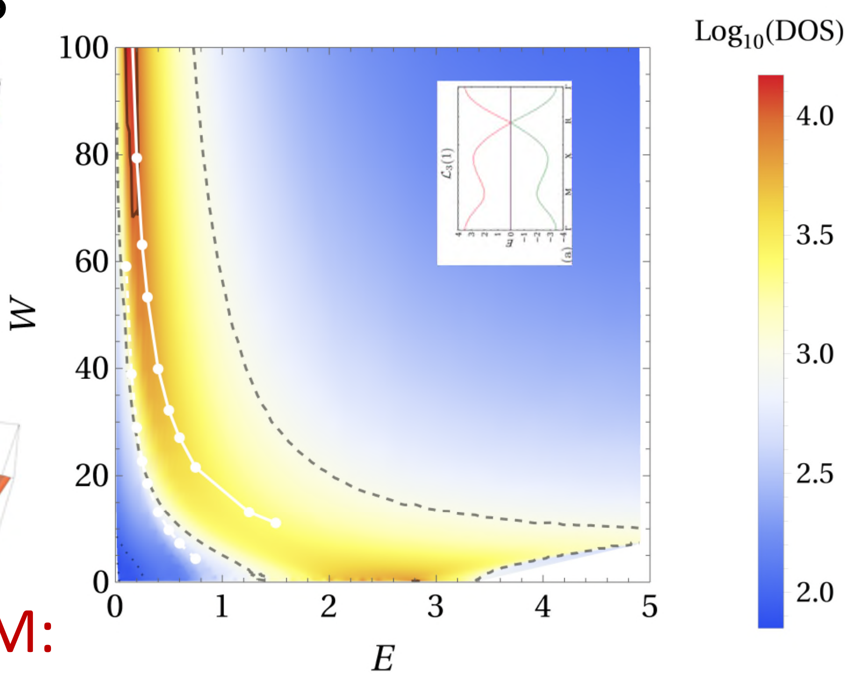
Extended Lieb models in 2 and 3D with CLS-preserving disorder

- DOS

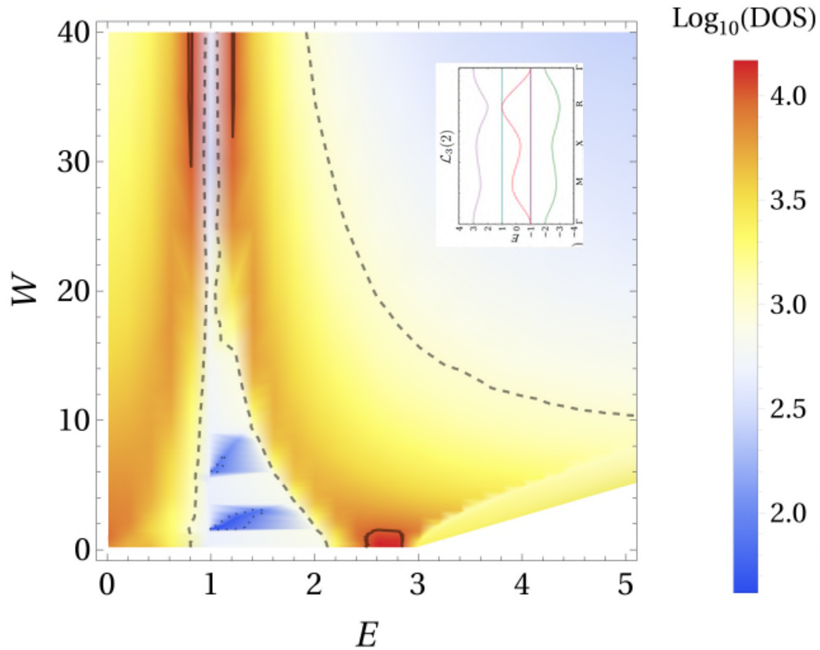
$\mathcal{L}_2(1)$

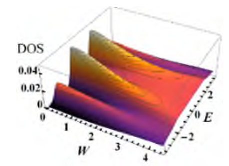


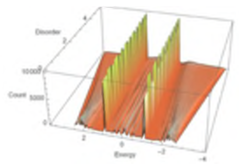




$\mathcal{L}_2(2)$







- TMM:
  - much harder since effectively less disorder on renormalized sites, hence harder to converge
- How to compute modified phase diagrams for CLS-preserving disorder?

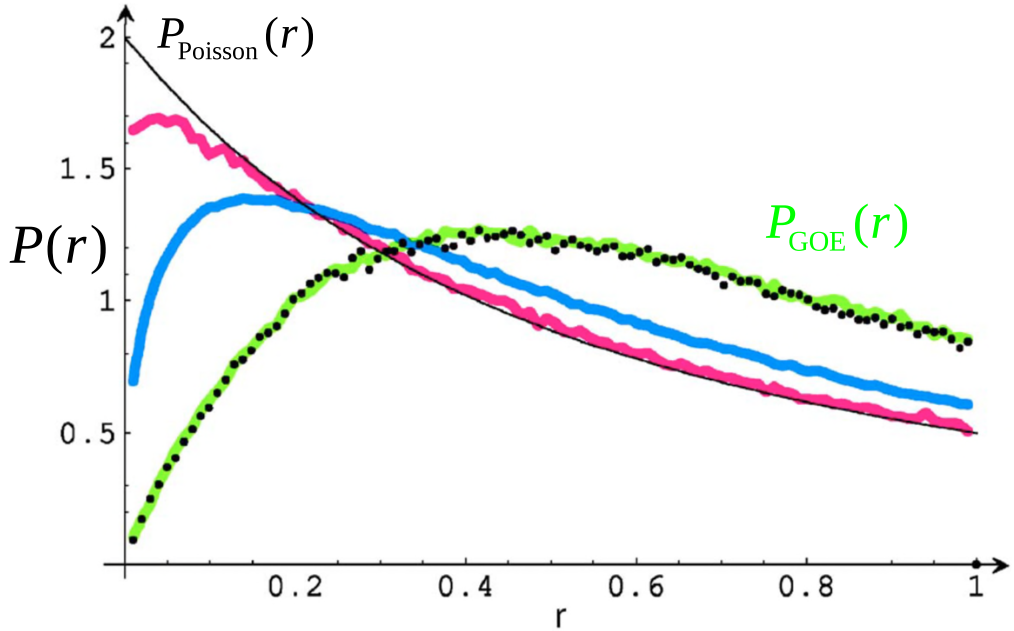
Energy-level ratio statistics (without unfolding)

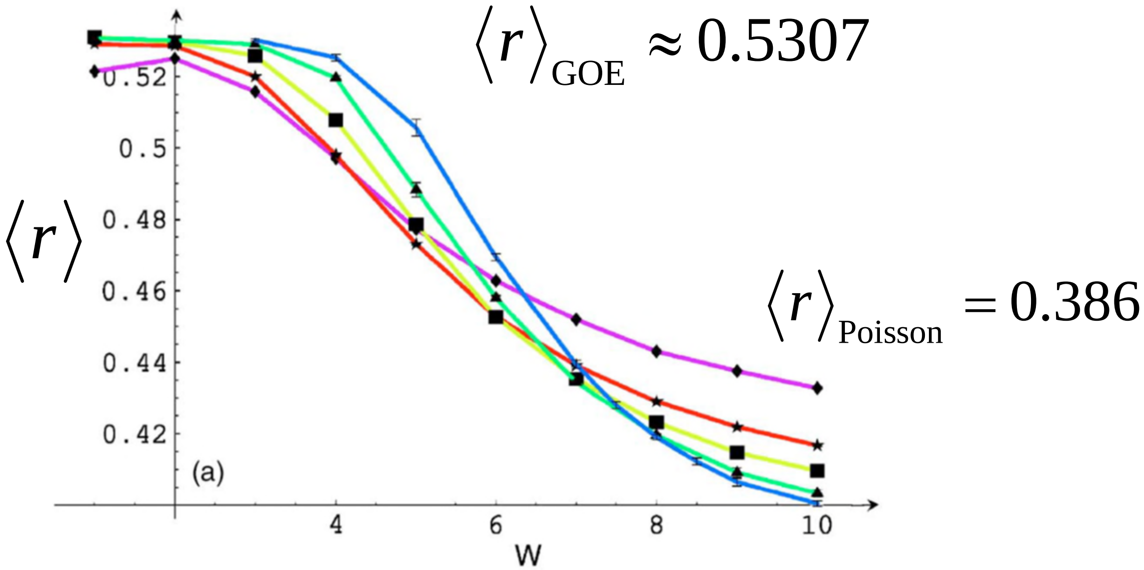
V. Oganesyan and D. A. Huse, Phys. Rev. B **75**, (2007):

$$0 \le r_n = \min\{s_n, s_{n-1}\} / \max\{s_n, s_{n-1}\} \le 1$$

$$\text{mean}: \langle r \rangle = \int_0^1 P(r) r \, \mathrm{d} r$$

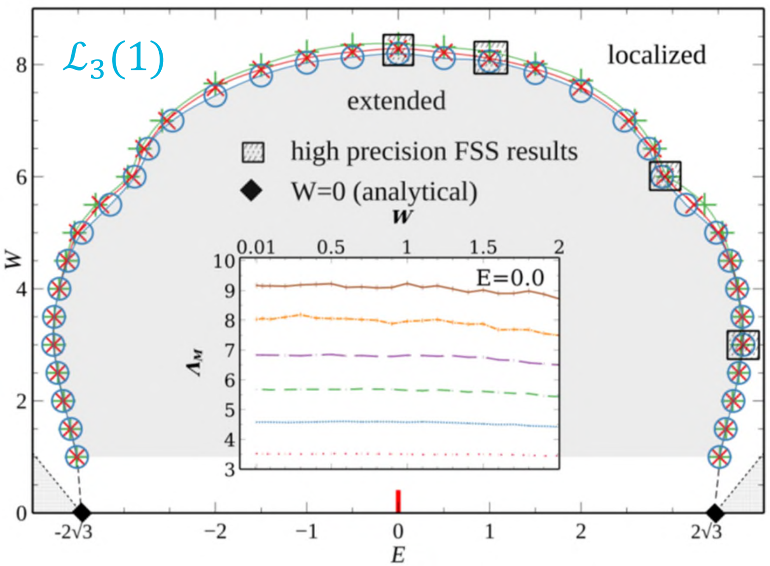
$$(s_n = E_n - E_{n-1})$$

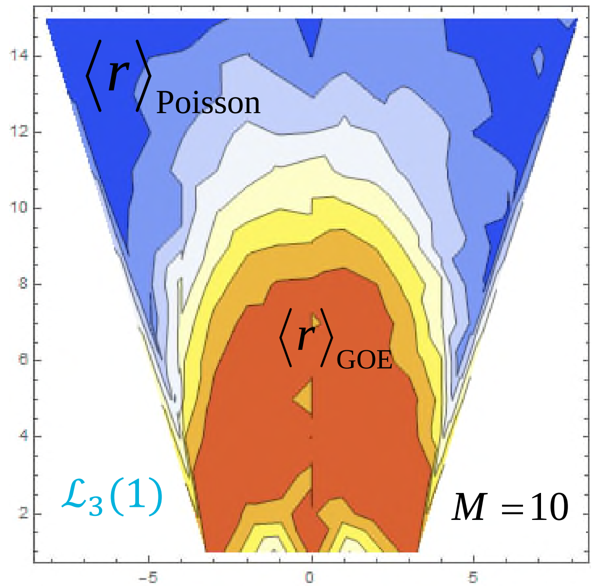


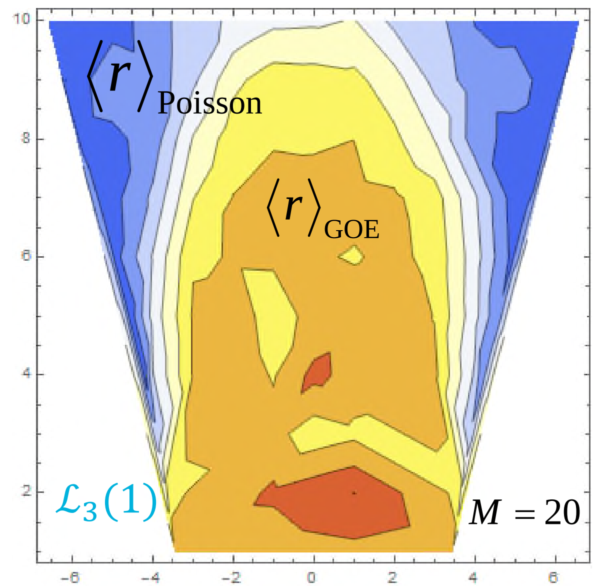


Does it work? Testing for the full disorder, equal on cube and Lieb

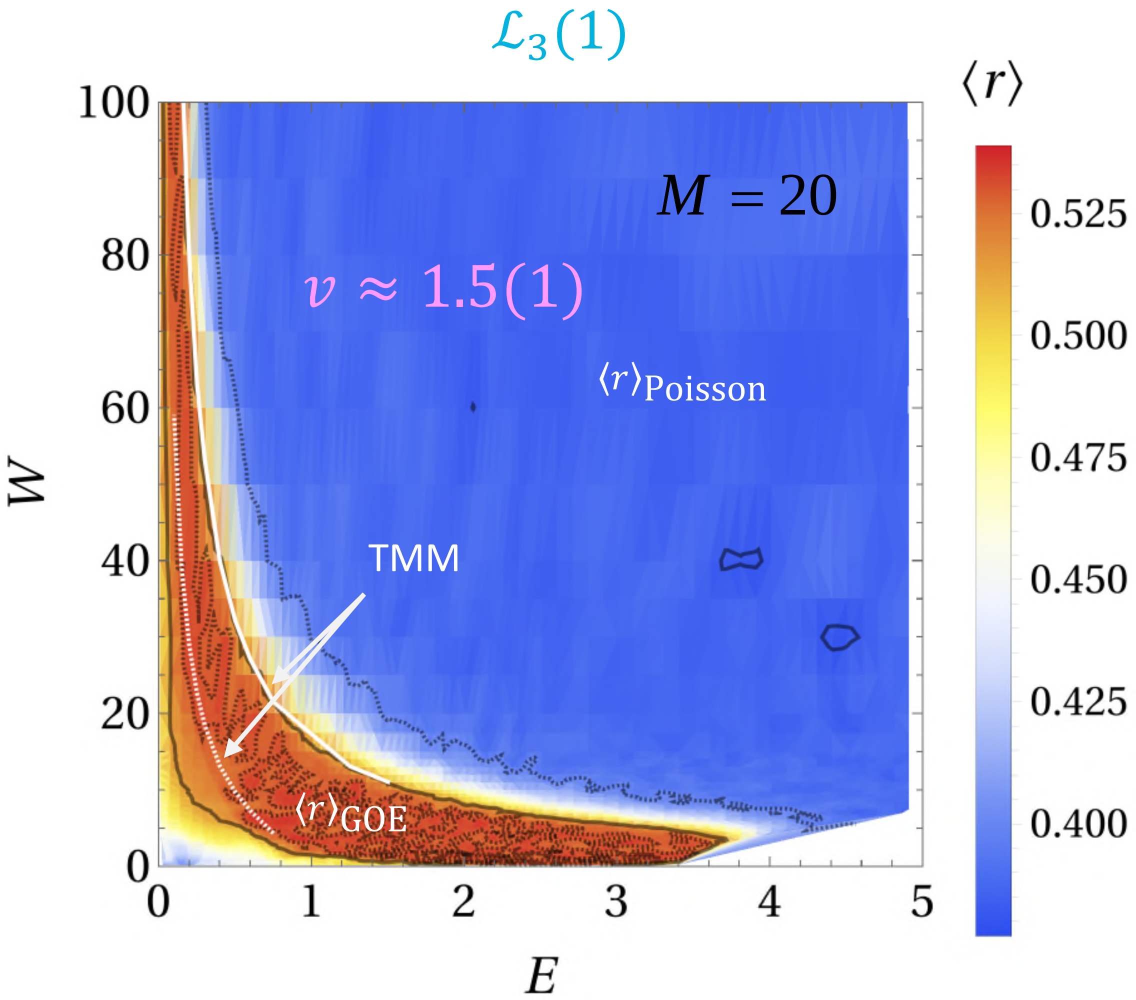
- TMM:
  - Phase boundaries determined from scaling behavior with small $M^2 = 6^2, 8^2, 10^2 = 36, 64, 100$ (with 99% target accuracy)
- Sparse-diagonalization
  - Phase boundaries determined from $\langle r \rangle$ for $M^3 = 10^3, 20^3$, i.e. sites
  $N = (3 \times 10)^3 = 27000, (3 \times 20)^3 = 216000$

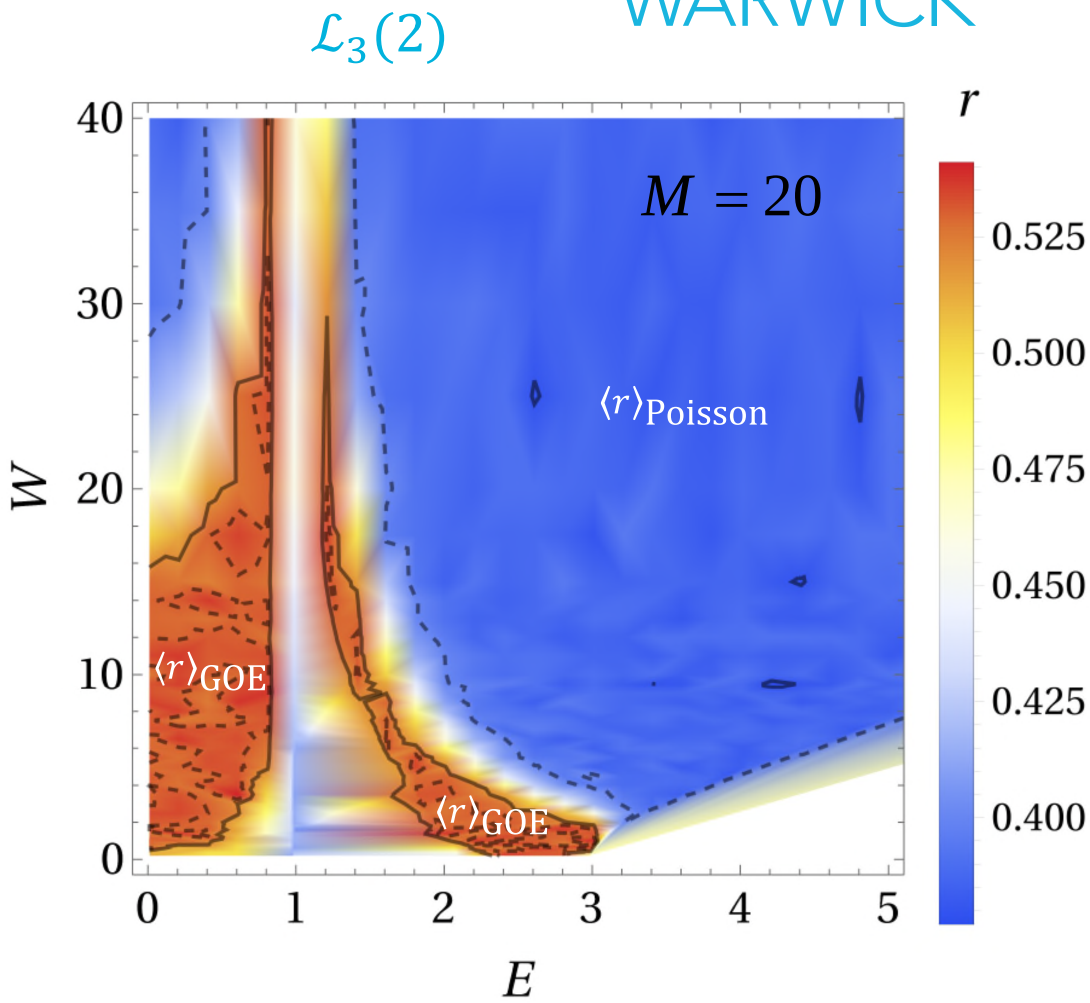




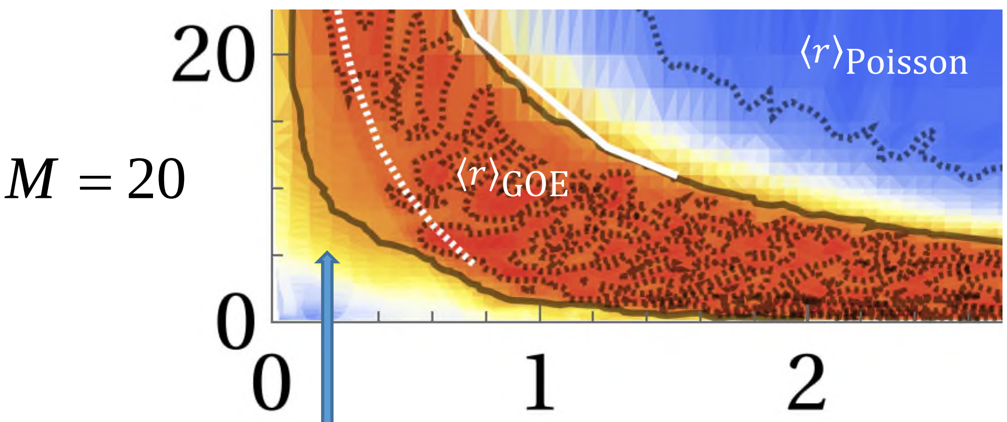


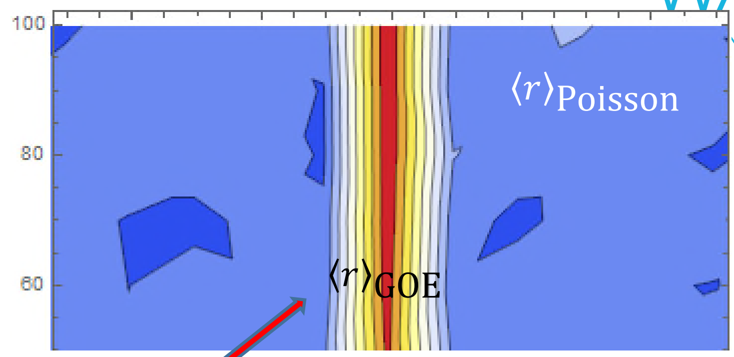
3D Lieb model with CLS-preserving disorder

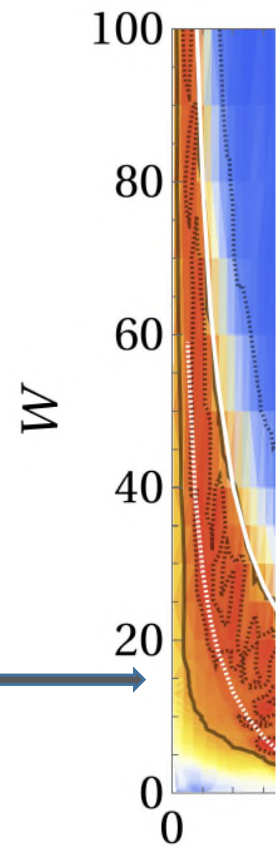




3D Lieb model with CLS-preserving disorder, 1st results



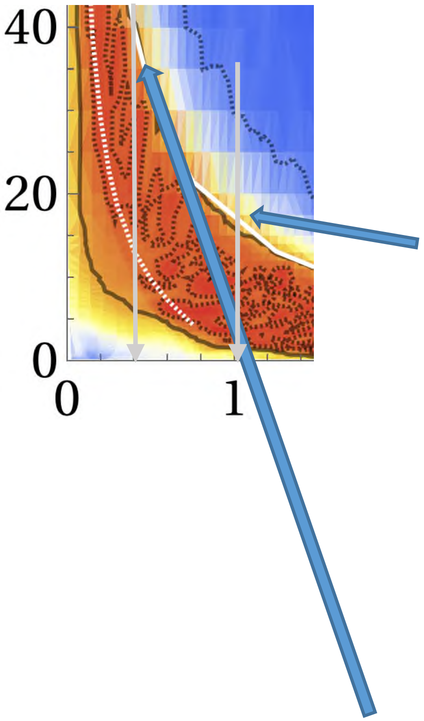




- "inverse" Anderson transition?
- CLS appear to show $\langle r \rangle$ values for GOE. Superposition of CLS!
- Non-CLS states delocalize close to FB energy *E*=0!?
- BORING? No more!

TMM:

$$M^2 = 16^2(< 0.1\%), 18^2, 20^2, \cdots, 26^2(< 0.5\%)$$

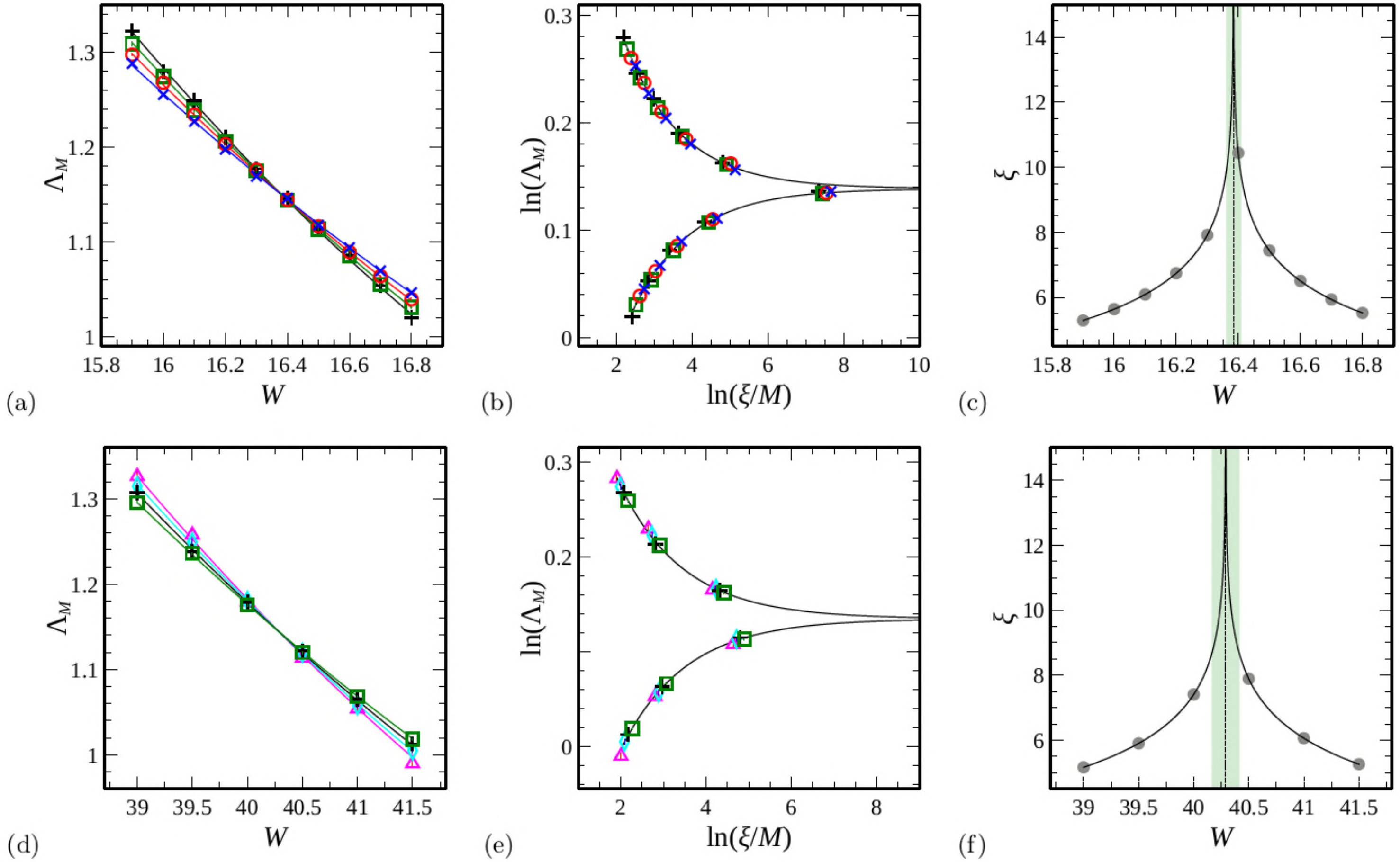


$E = 1$

$\nu = 1.51(5)$

$E = 0.4$

$\nu = 1.51(4)$



$\langle r \rangle + \langle z \rangle$-values:

$N^3 = 16^3, 18^3, 20^3, \cdots, 24^3, L = 4 \times N^3$

10000 samples for each disorder value

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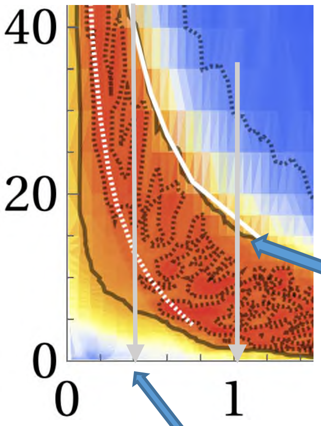


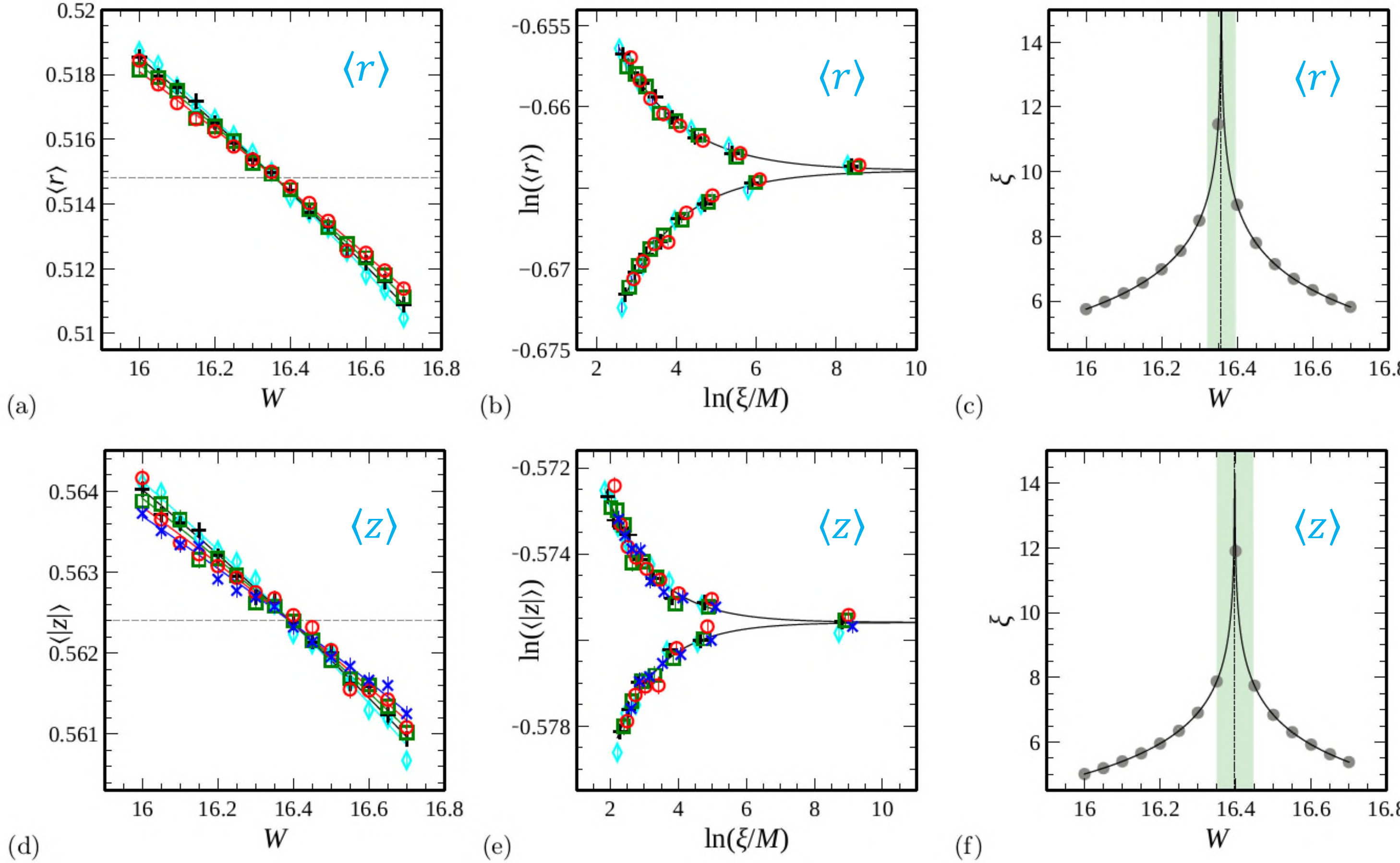
$E = 1$

$\nu = 1.54(9)$

$E = 1$

$\nu = 1.44(10)$

$$|z_n| = \frac{|E_n - E_{NN}|}{|E_n - E_{NNN}|}$$



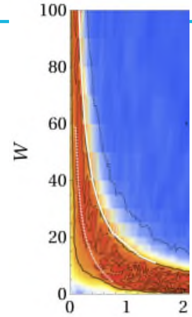
Enhancement of CLS stability

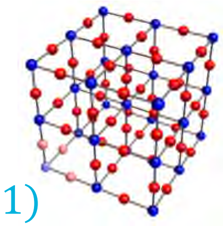
- Projected wave function

$$\sum_{\boldsymbol{r}\in \text{all } N^3} |\psi_E(\boldsymbol{r})|^2 = 1$$

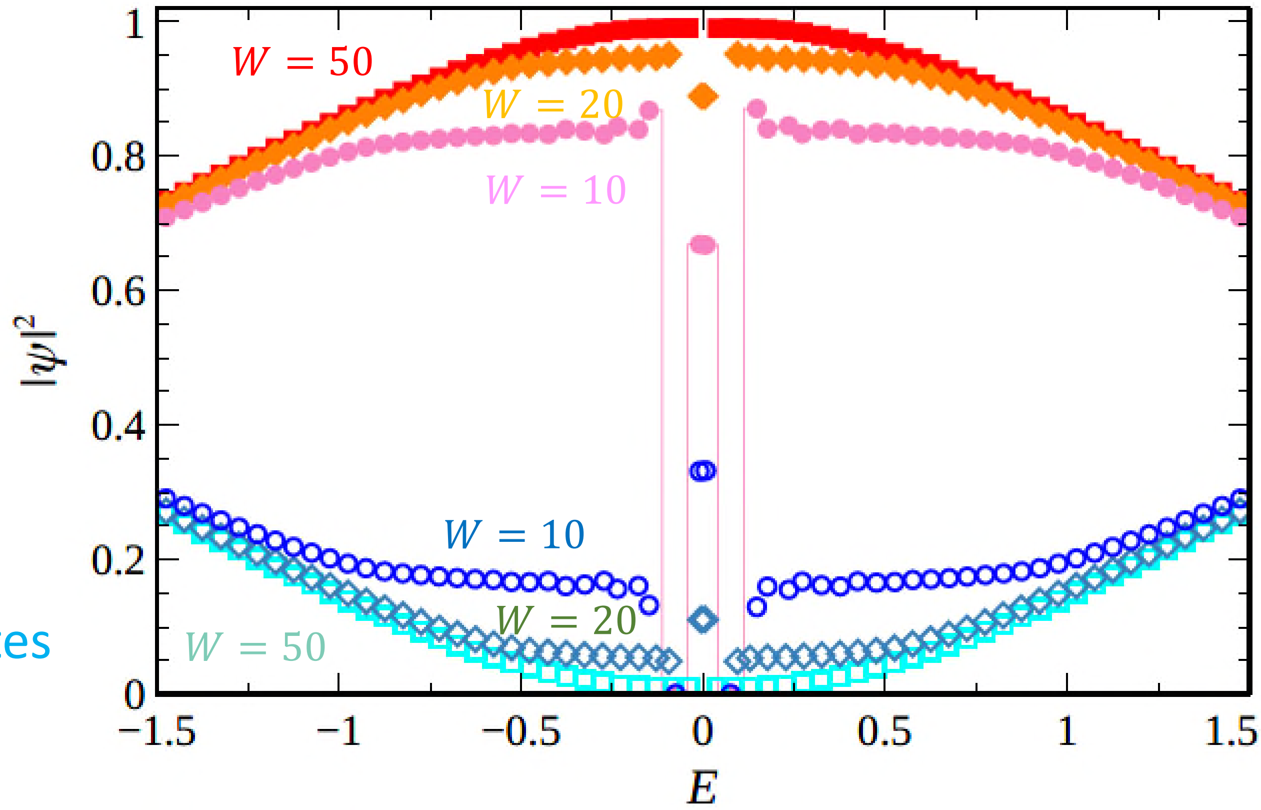
$$\sum_{\boldsymbol{r}\in \text{Lieb}} |\psi_E(\boldsymbol{r})|^2, \quad \sum_{\boldsymbol{r}\in \text{cube}} |\psi_E(\boldsymbol{r})|^2$$

- Close to CLS at $E = 0$, disorder moves dispersive states
  - ➢ Towards $E = 0$ when on Lieb sites
  - ➢ Away from $E = 0$ when on cube sites



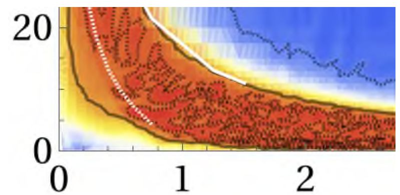


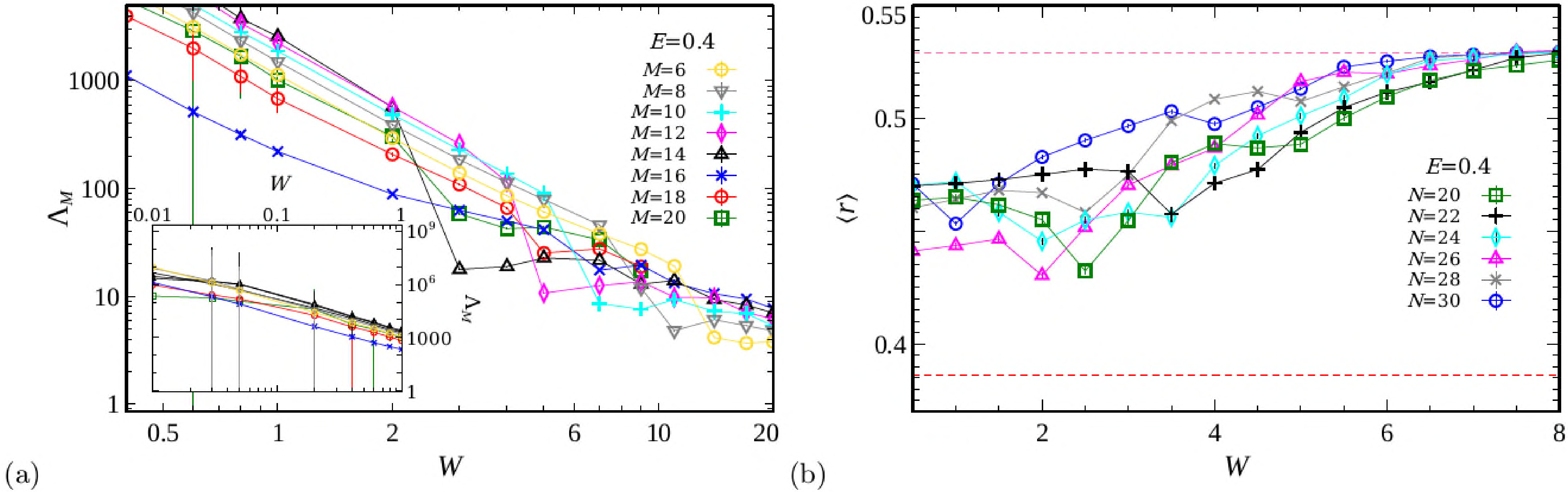
$\mathcal{L}_3(1)$



No “inverse Anderson” transition – something else

- TMM or ELS or wave functions (P) cannot identify a single, system-size independent crossing point
- So either our system sizes $M^2 \times \infty$ (TMM) up to $M = 20$ or N^3 (ELS, WF) are too small or this is not a “transition”, but a more gentle crossover!





ELS for quasi-periodic tilings

$$H = \sum_{\langle \boldsymbol{r} \neq \boldsymbol{r}' \rangle} t_{\boldsymbol{r},\boldsymbol{r}'} |\boldsymbol{r}\rangle\langle \boldsymbol{r}'|$$

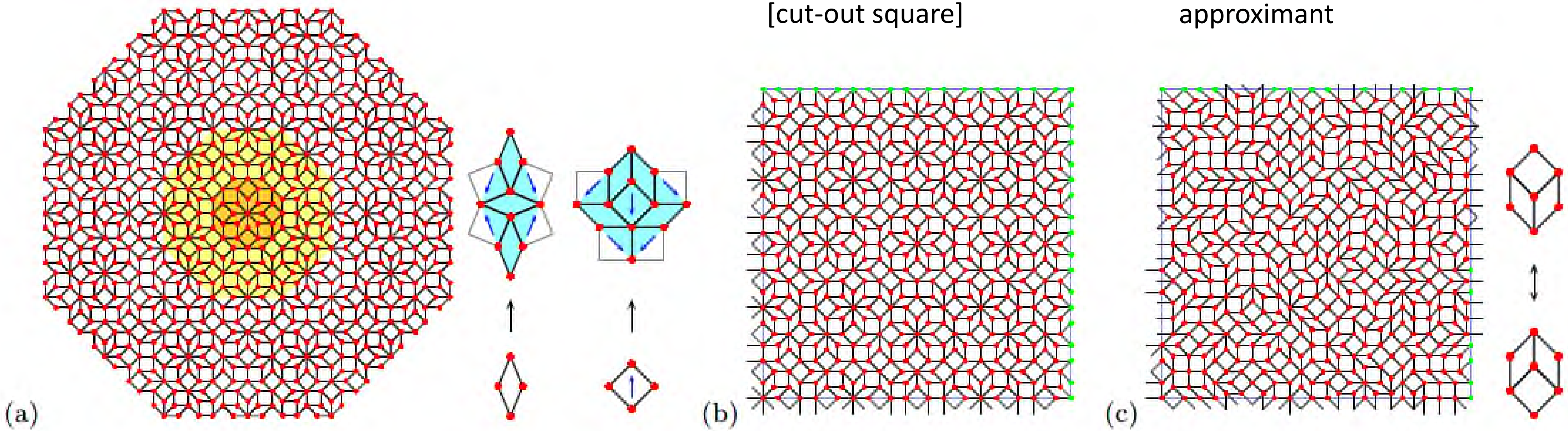
- cover space without gaps or overlaps but never repeat exactly.
- local regularity with global aperiodicity
- wavefunctions shows scale-invariant fluctuations across the tiling.

We studied Ammann-Beenker tilings up to 157369 vertices and tilings with random “phase” flips including up to up to 8358000 *r*-values.

AB tiling, inflation 3

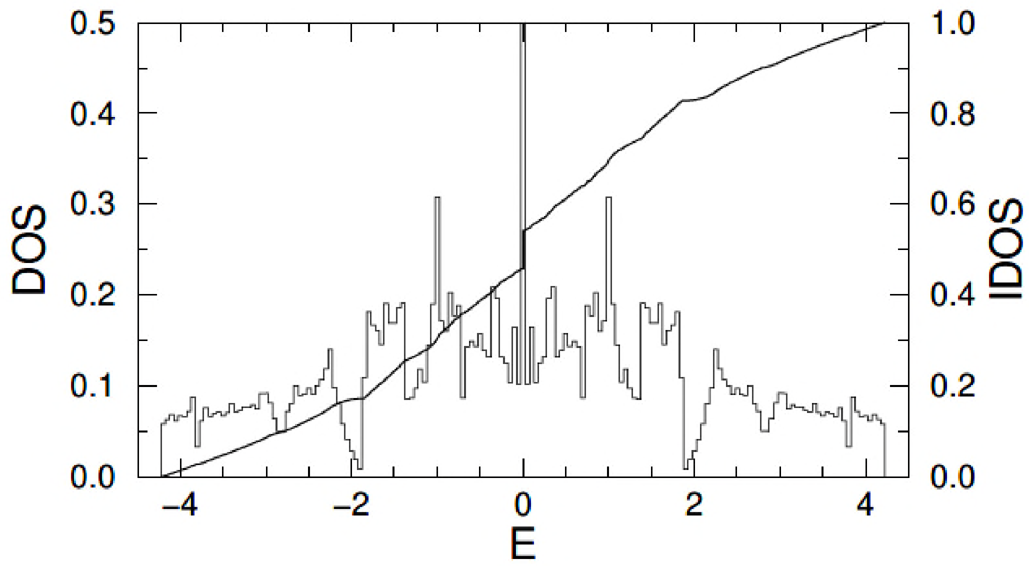
Periodic approximant [cut-out square]

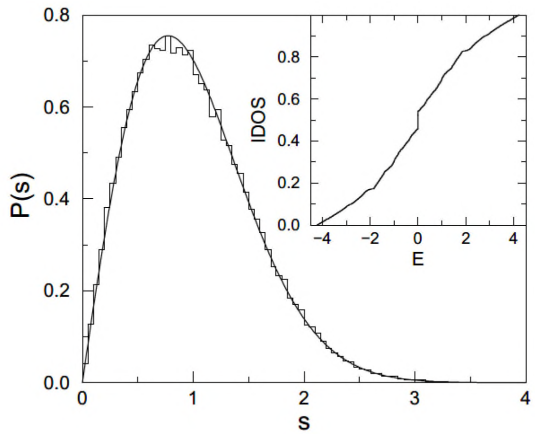
Phase-flipped AB approximant

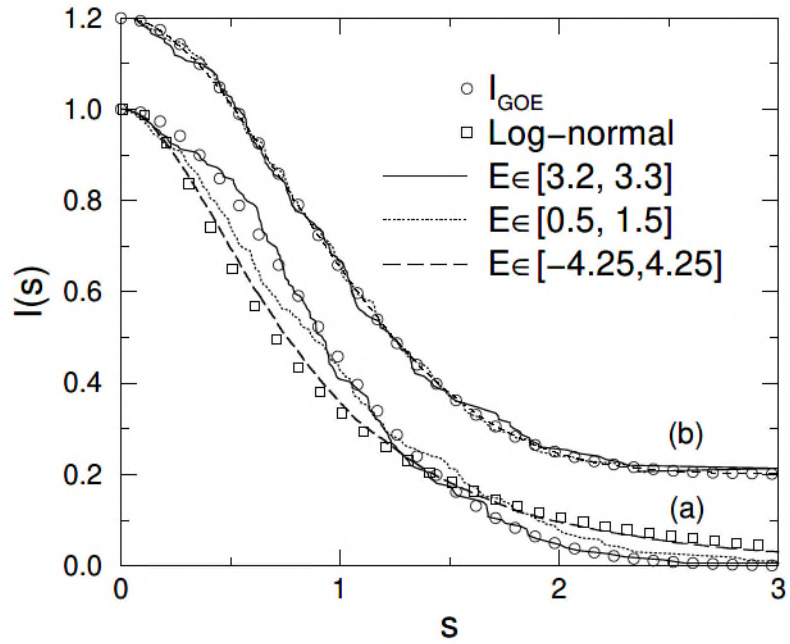


ELS for quasi-periodic tilings

- DOS is spiky, needs “unfolding”
- with unfolding, universal results are possible
- without unfolding, results are not expected to be universal
- we do unfolding”:







ELS is GOE

- when taking into account dihedral symmetry, then 7 independent sectors
- all sectors follow

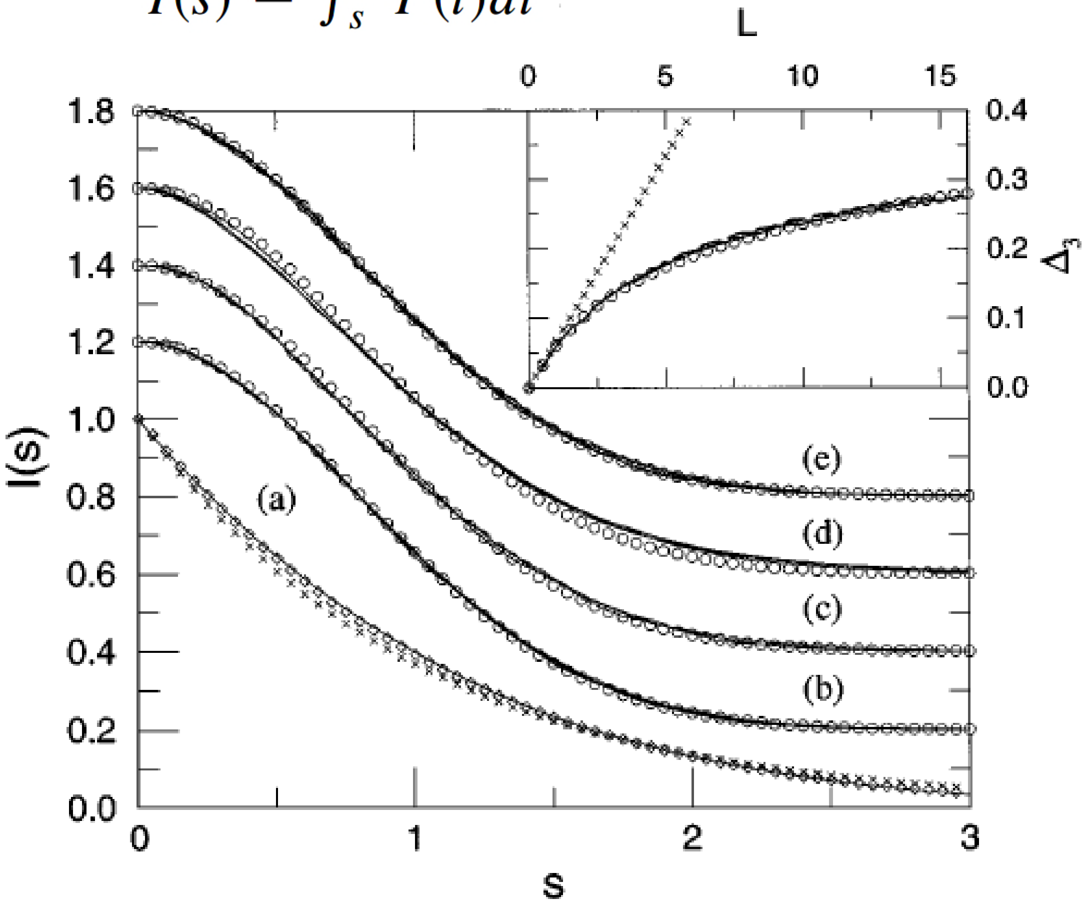
$$P_{\mathrm{W}}^{(k)}(s) = \frac{d^2}{ds^2}\left[\mathrm{erfc}\left(\frac{\sqrt{\pi}}{2}\frac{s}{k}\right)\right]^k$$

- individual sectors follow

$$P_W(s) = \frac{\pi s}{2}\exp(-\pi s^2/4) \approx P_{GOE}(s)$$

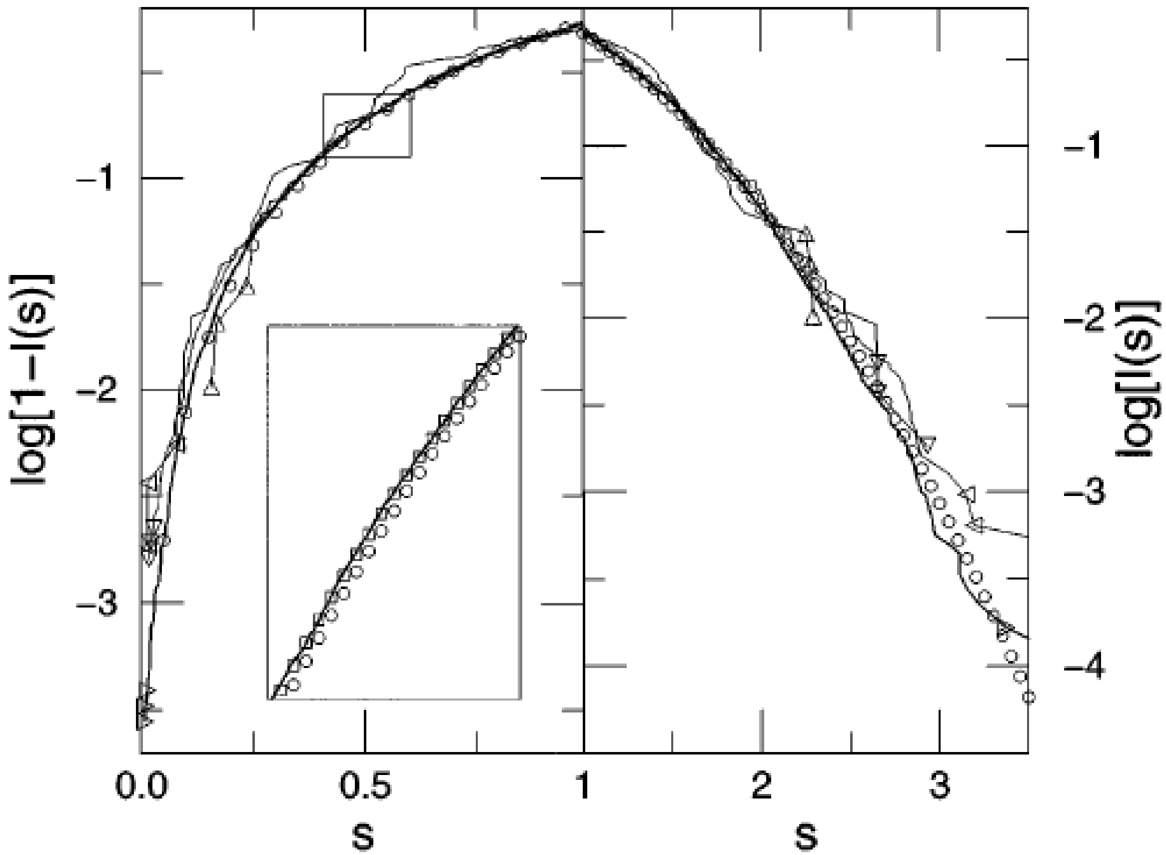
- also works for other tilings and different “cuts” (Sinai-billiard shape) of AB tiling

$$I(s) = \int_s^\infty P(t)dt$$



ELS is GOE, better than Wigner!

- in fact, GOE is followed even better than Wigner surmise
- see small-*s* and large-*s* behaviour for the largest patch of one irreducible sector
- not often that “disordered” systems using the “standard model” show this



J. X. Zhong, U. Grimm, RAR, M. Schreiber, Phys. Rev. Lett. 80, 3996-3999 (1998).

ELS is GOE in QP tilings

maybe unfolding did this (we checked already then)?

- no, using *r*-value statistics, i.e. without unfolding gives same result: GOE
- 4887638 *r*-values used

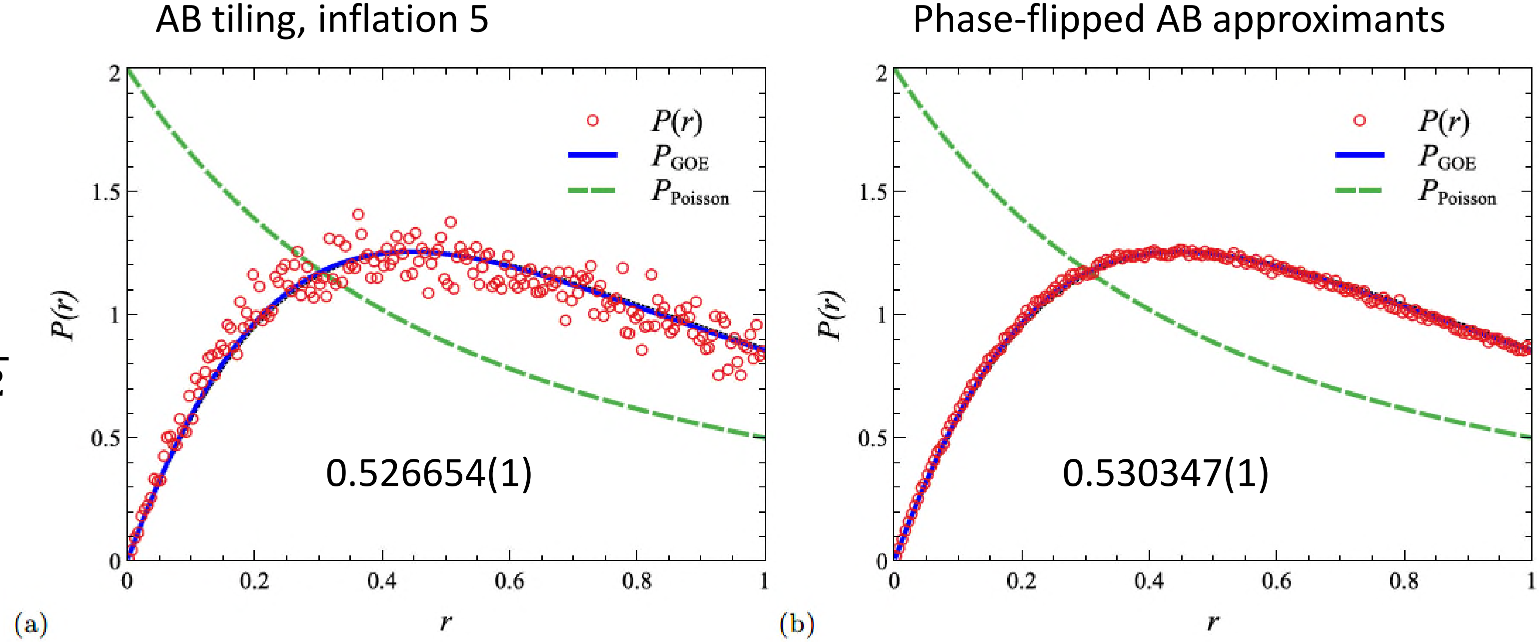
U. Grimm, RAR, Phys. Rev. B 104(6), L060201 (2021)

$$\langle r \rangle_{\text{GOE}} \approx 0.5307$$

$$\langle r \rangle_{\text{Poisson}} = 0.386$$

$$P_W(r) = \frac{27(r + r^2)}{4(1 + r + r^2)^{5/2}}$$

$$P_P(r) = \frac{2}{(1 + r)^2}$$



Interactions in the presence of disorder

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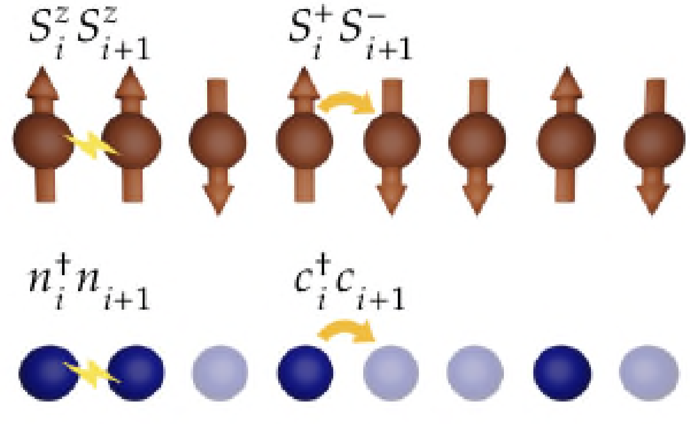
- Interactions mix all the (Fock) states, leading to ergodicity
- disorder leads to localized states, can that be strong enough to prevent mixing, hence absence of ergodicity?
- we are interested in all states, so not just a ground state question

- poster child/model (1D):

$$H = J \sum_{\langle ij \rangle}^{L} \left(S_i^x S_j^x + S_i^y S_j^y \right) + J_z \sum_{\langle ij \rangle} S_i^z S_j^z + \sum_{i}^{L} h_i S_i^z$$

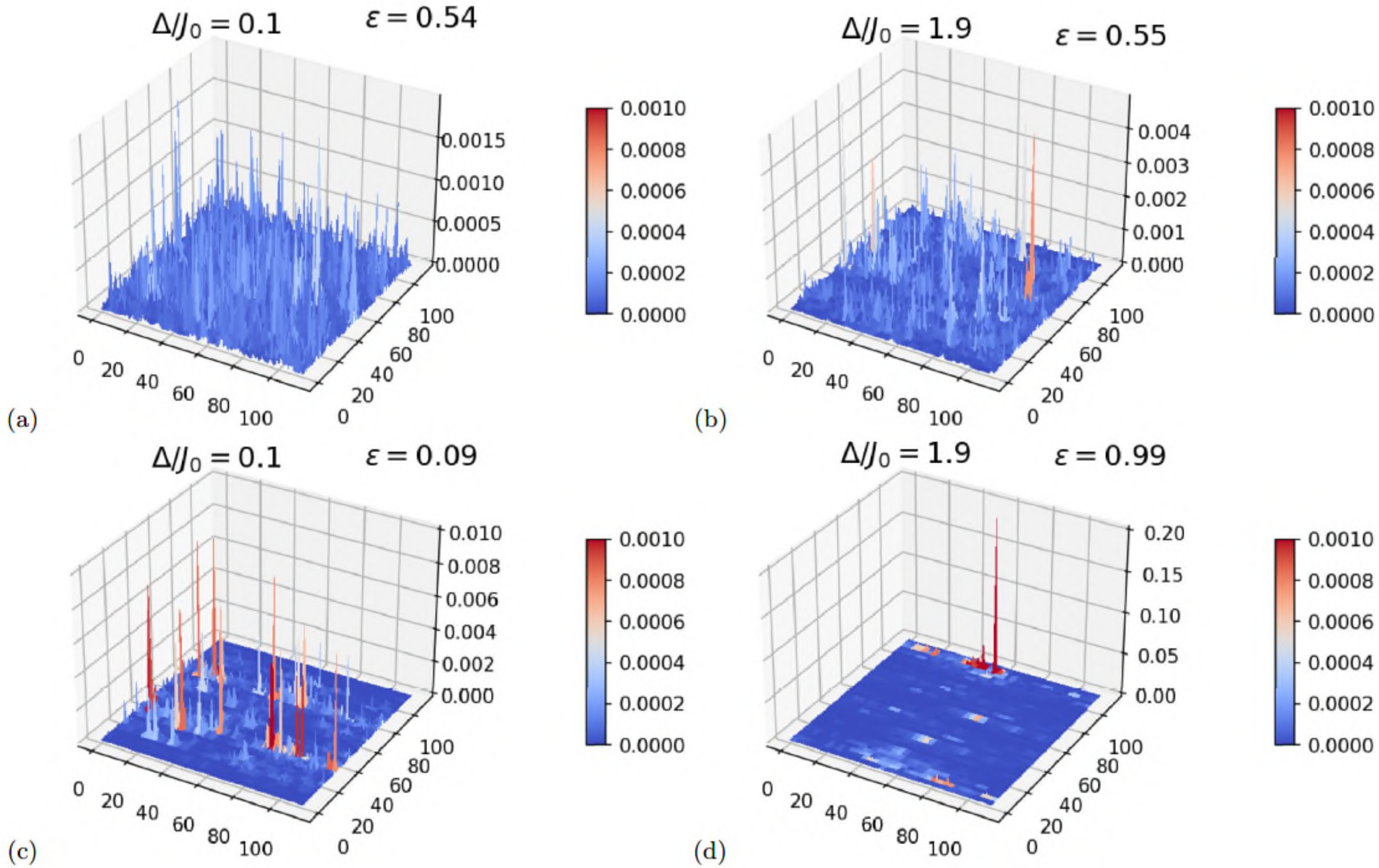
$$H = J \sum_{\langle ij \rangle}^{L} \left(c_i^{+} c_j + h.c. \right) + J_z \sum_{\langle ij \rangle} n_i \; n_j + \sum_{i}^{L} h_i n_i$$



A **picture** of MLB:

- L=16, 12870 states in Fock space (in *m*=0 sector)
- roughly equal to $12996=144^2$
- increasing the disorder leads to less Fock states contributing to eigenstate -> Fock-space localization

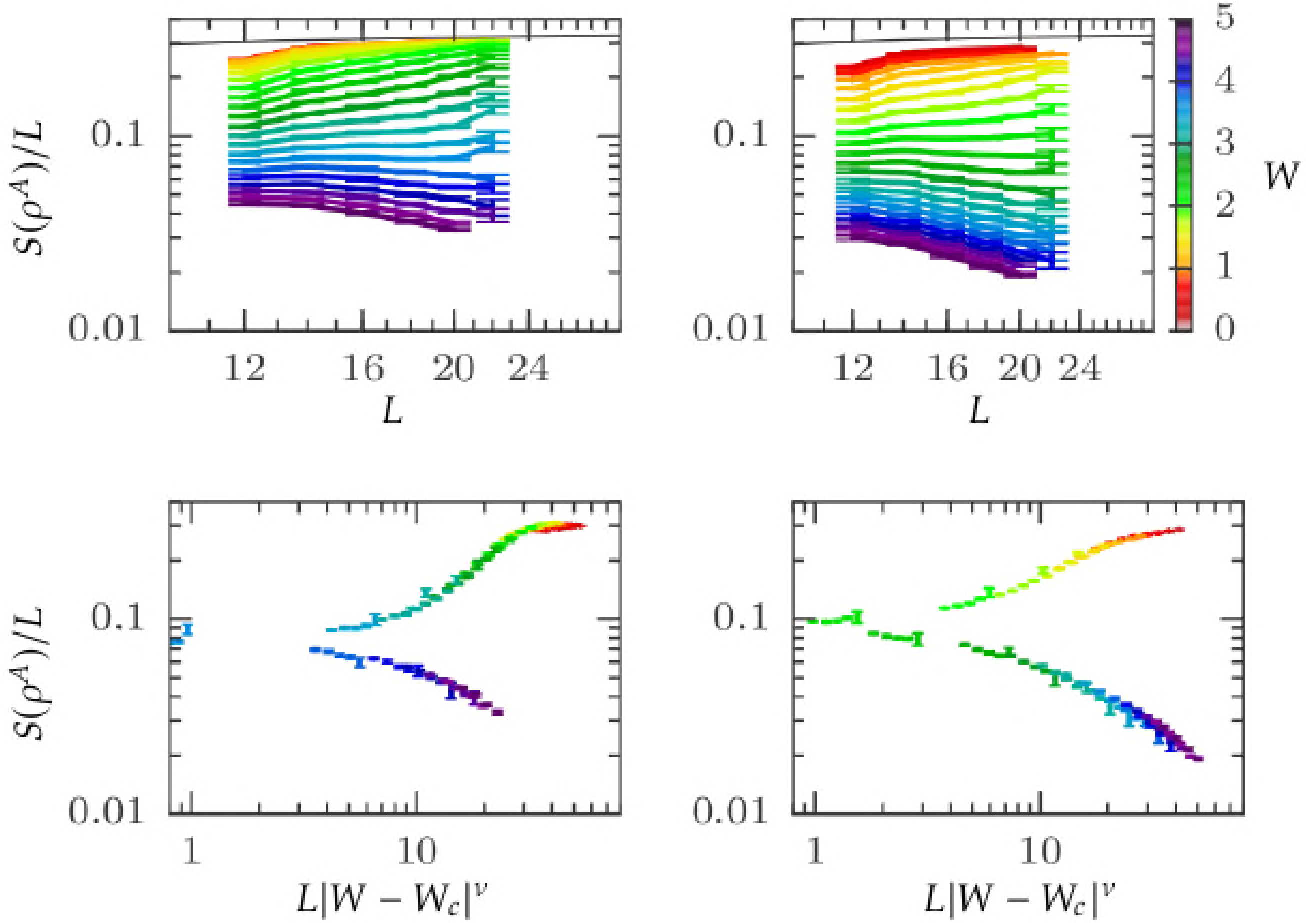
centre of spectrum



bottom of spectrum

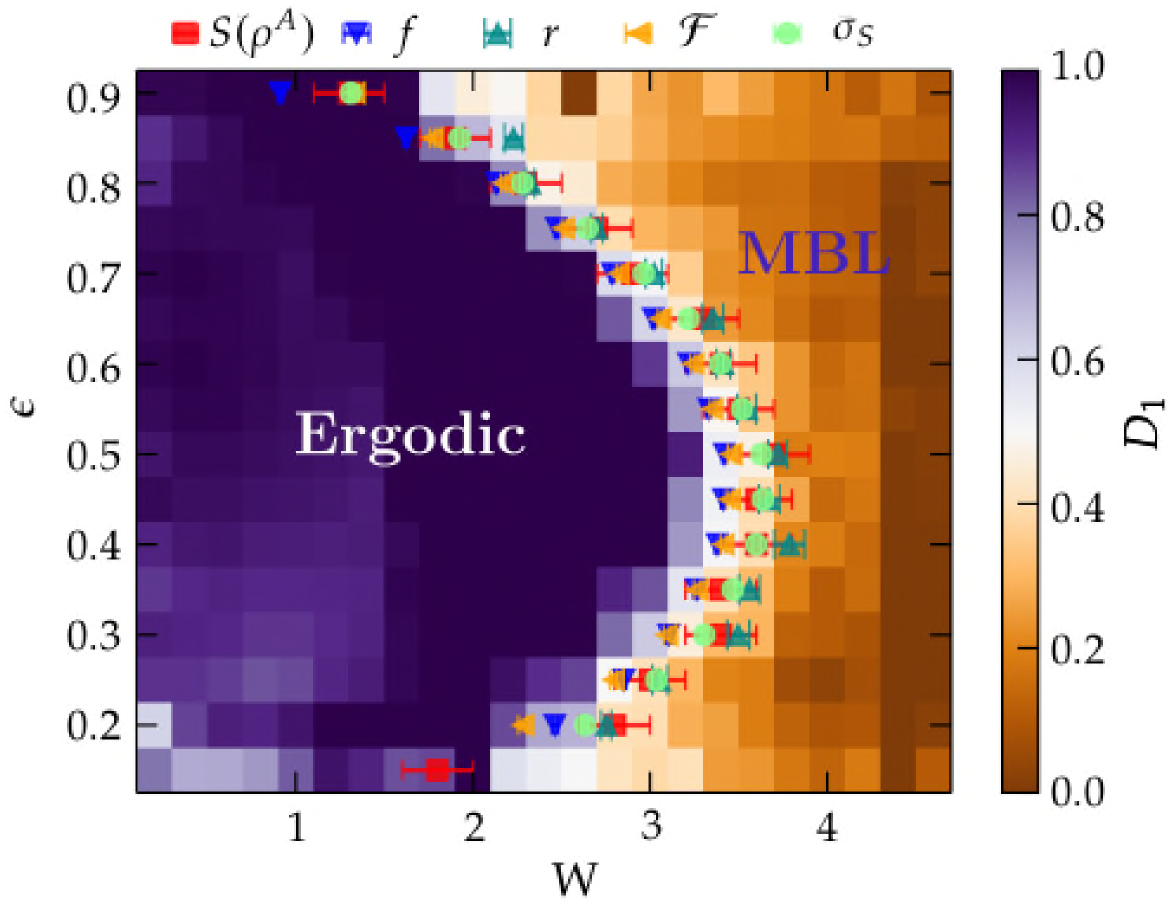
Finite-Size Scaling hints at a transition:

- Luitz, D. J., N. Laflorencie, and F. Alet (2015). “Many-body localization edge in the random-field Heisenberg chain”. In: Physical Review B 91.8, p. 81103.



Even a full phase diagram can be computed:

- entanglement-based measure
- Fidelity-based measures
- r-value
- MBL = many-body localization



Interactions in the presence of disorder

$$H = \sum_{\langle ij \rangle}^{L} J_{ij}\left(c_i^{+} c_j \; + h.c.\right) + 1 \sum_{\langle ij \rangle} n_i \; n_j$$

$$H = \sum_{\langle ij \rangle}^{L} J_{ij}\left(S_i^x S_j^x + S_i^y S_j^y\right) + 1 \sum_{\langle ij \rangle} S_i^z S_j^z$$

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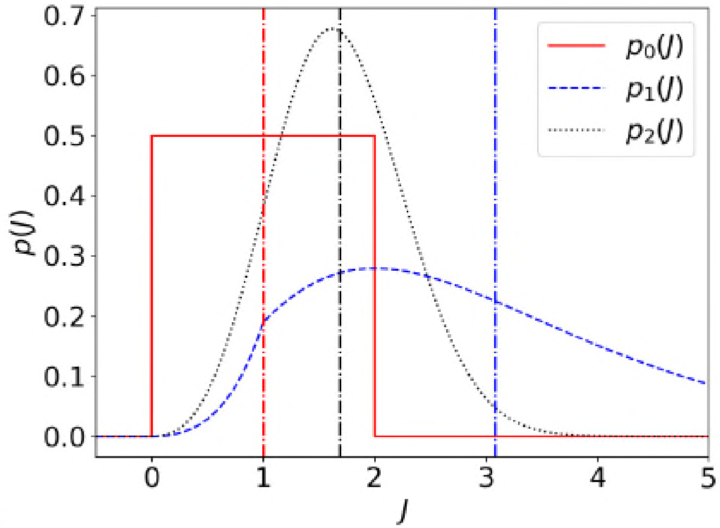
- random exchange model, retains full SU(2) symmetry
- Vasseur et al. [PRB 93, 134207 (2016)] **report transition** from ergodic states to MBL states at strong disorders.
- Protopopov et al. [PRX 10, 011025 (2020)] find **states intermediate** between extended states and MBL states even at strong disorders, i.e. **no MBL transition**
- Siegl and Schliemann (level statistics) argue [NJP 25, 123002 (2023)] **for transition** from ergodic phase to a **phase that is different** from both ergodic and MBL
- Saraidaris et al. [PRB 109, 094201 (2024)] suggest that thermalization and **delocalization appear** at large system sizes, L = 48, using tDMRG
- Han et al. [arXiv:2411.09368] found **no evidence of an MBL transition** studying the time and disorder dependence of multifractal exponents

-> a complicated model!

- **ground state properties** real-space renormalize to random singlet phase with increasing disorder:

Ma, Dasgupta, Hu, PRL 43, 1434 (1979); Dasgupta, Ma, PRB 22, 1305 (1980); Fisher, PRB 50, 3799 (1994).

- *P(J)* peaked at J=0 with long-tail for *J*>0
- Let's look at more *P(J)*'s
- Assure that $J_{ij} > 0$ for all *i,j* to stay *antiferromagnetic*



(a)

[200] "Spectral and Entanglement Properties of the Random Exchange Heisenberg Chain", Y. Gao, RAR, Phys. Rev. B 111, 104202 (2025)

SU(2)-invariant, i.e. S^2 and S_z conserved (quantum numbers *s*,*m*)

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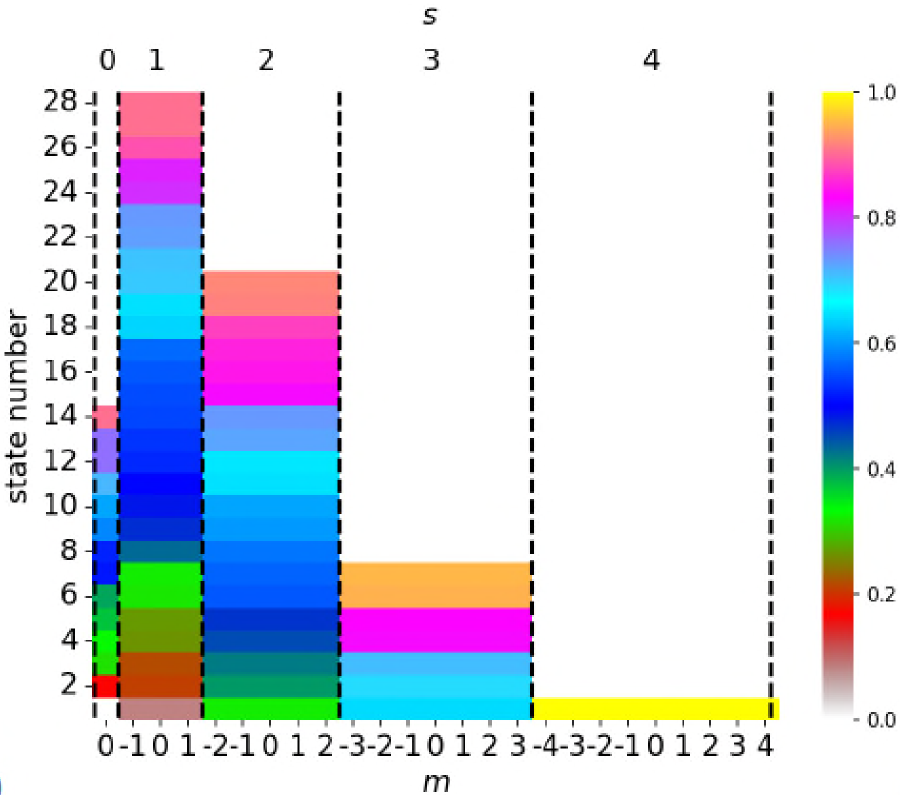
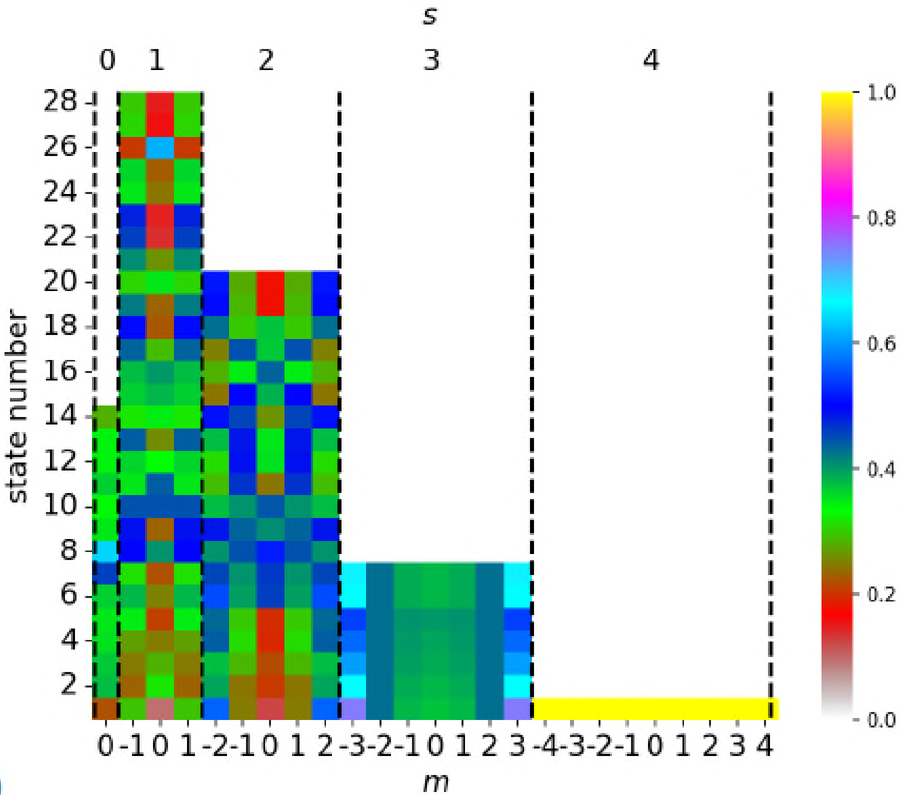
energy



(a)

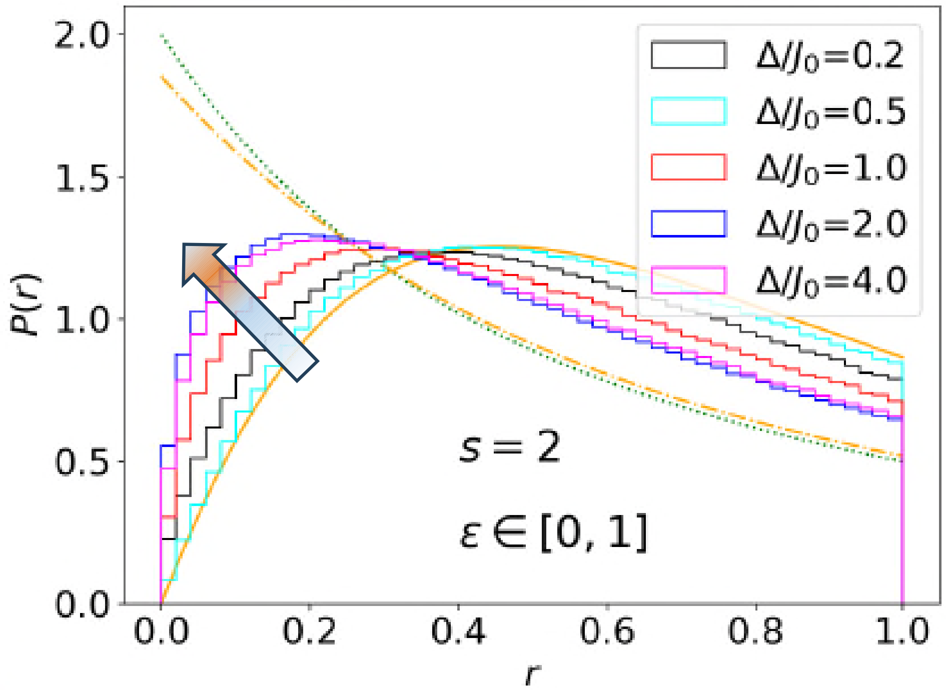
participation ratio



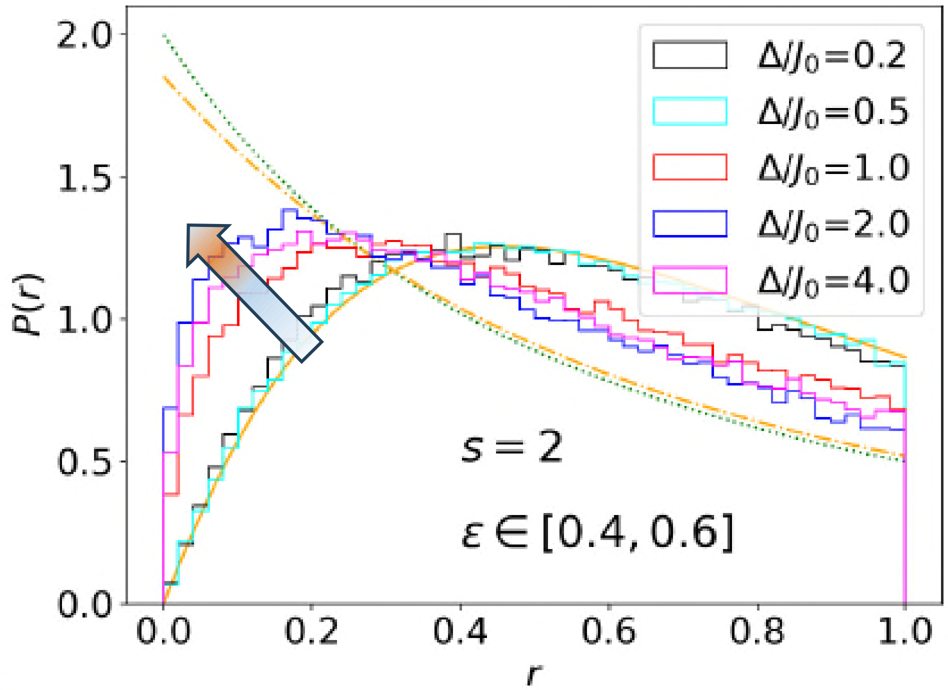
(b)

r-value distributions

- full spectrum:
- centre of spectrum:



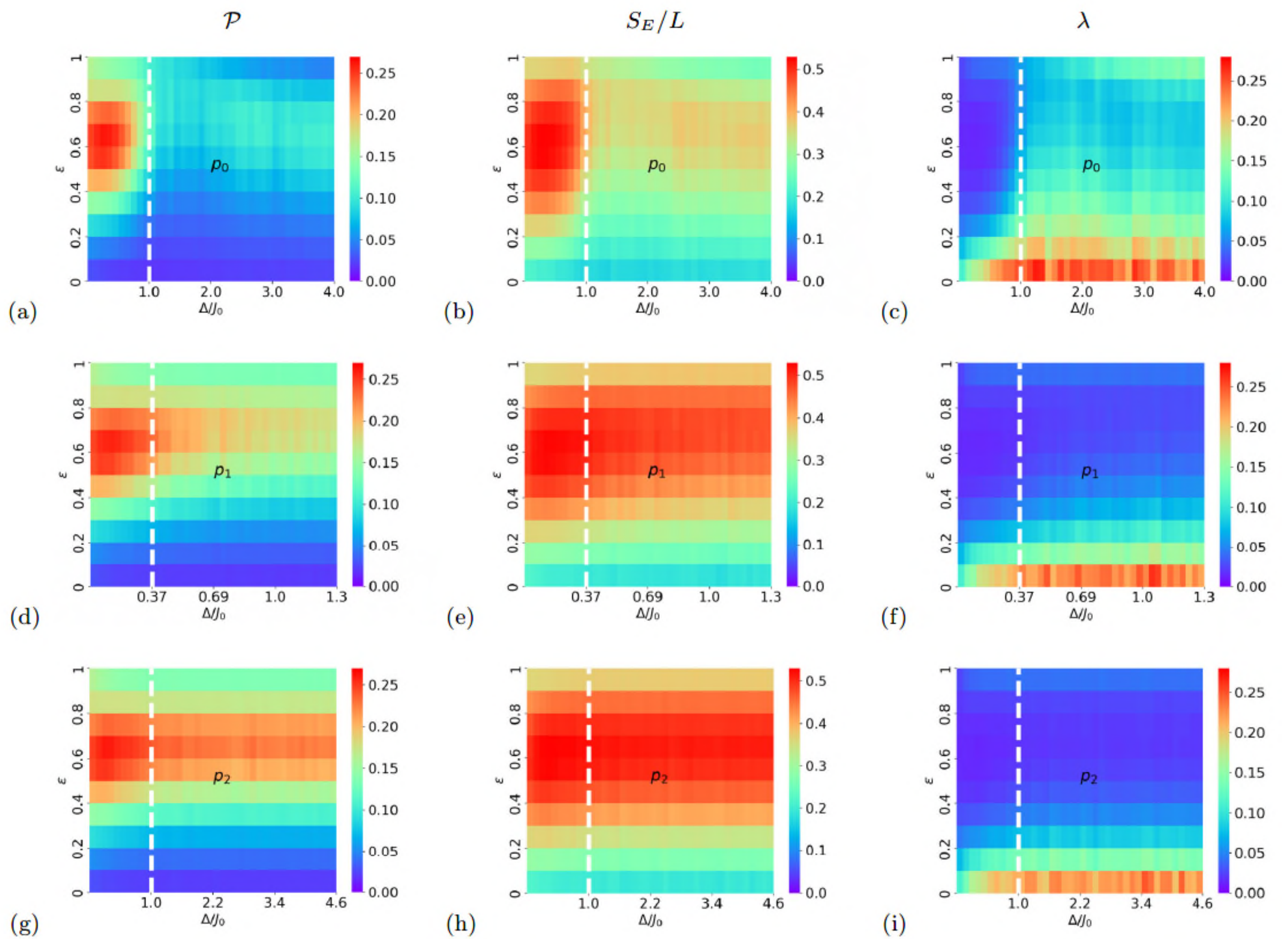
(a)



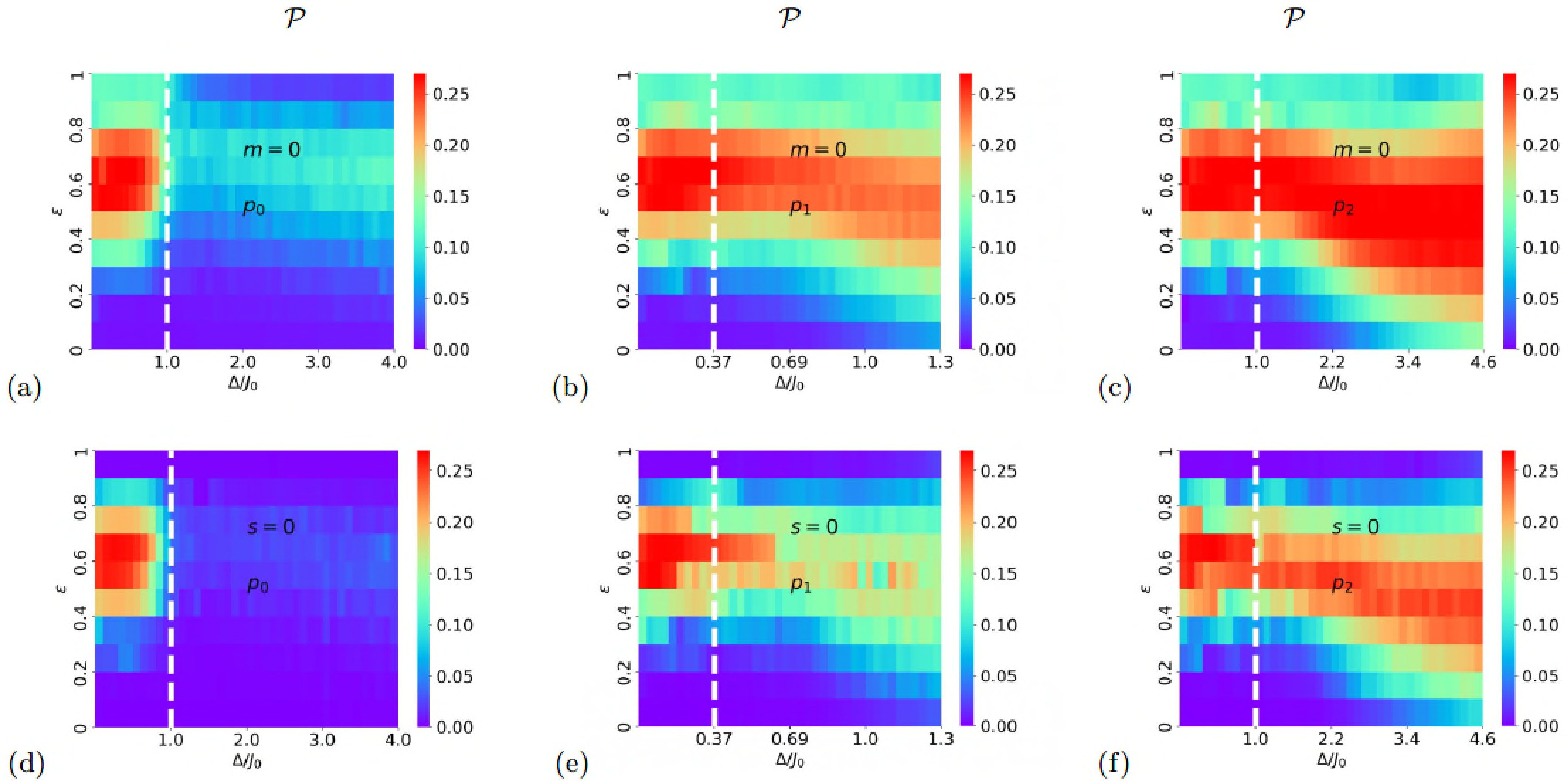
(b)

Different *P(J)* give different answers (state-based measures)

- transition when FM couplings become important?
- weak transition?
- no transition?

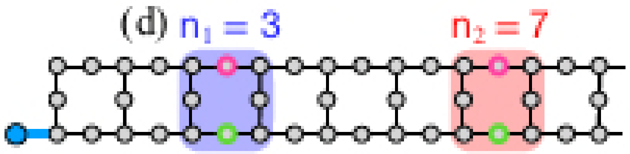


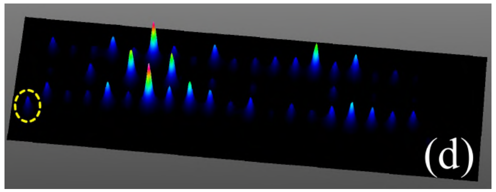
Different *sectors* give different answers



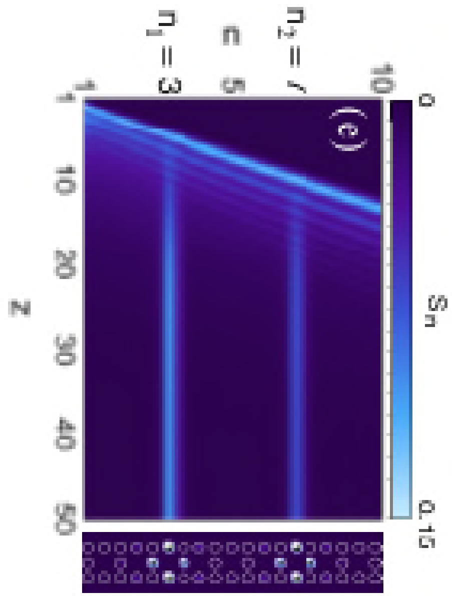
Conclusions/summary/outlook

- Flat band systems, CLS, can be used for storage

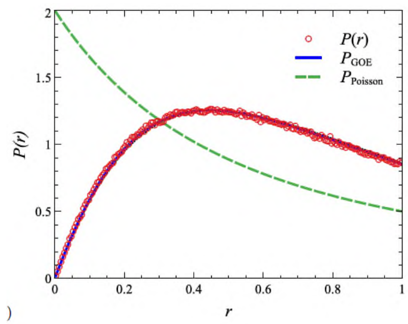




"Quantum storage with flat bands“, C. Danieli, J. Liu, RAR, R. A. Vicencio, arXiv:2508.01846



- Quasi-period systems show GOE



- Many-body localization uses ELS

Thanks for your attention!

These results are contained in

X. Mao, J. Liu, J. Zhong, and R. A. Römer, Phys. E Low-Dimensional Syst. Nanostructures **124**, 114340 (2020).

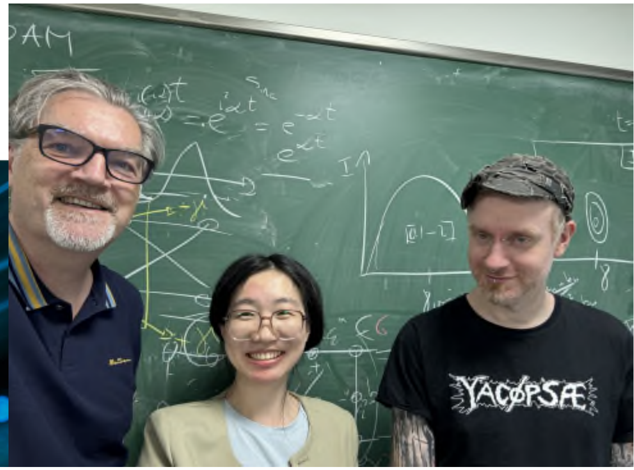
J. Liu, X. Mao, J. Zhong, and R. A. Römer, Phys. Rev. B **102**, 174207 (2020).

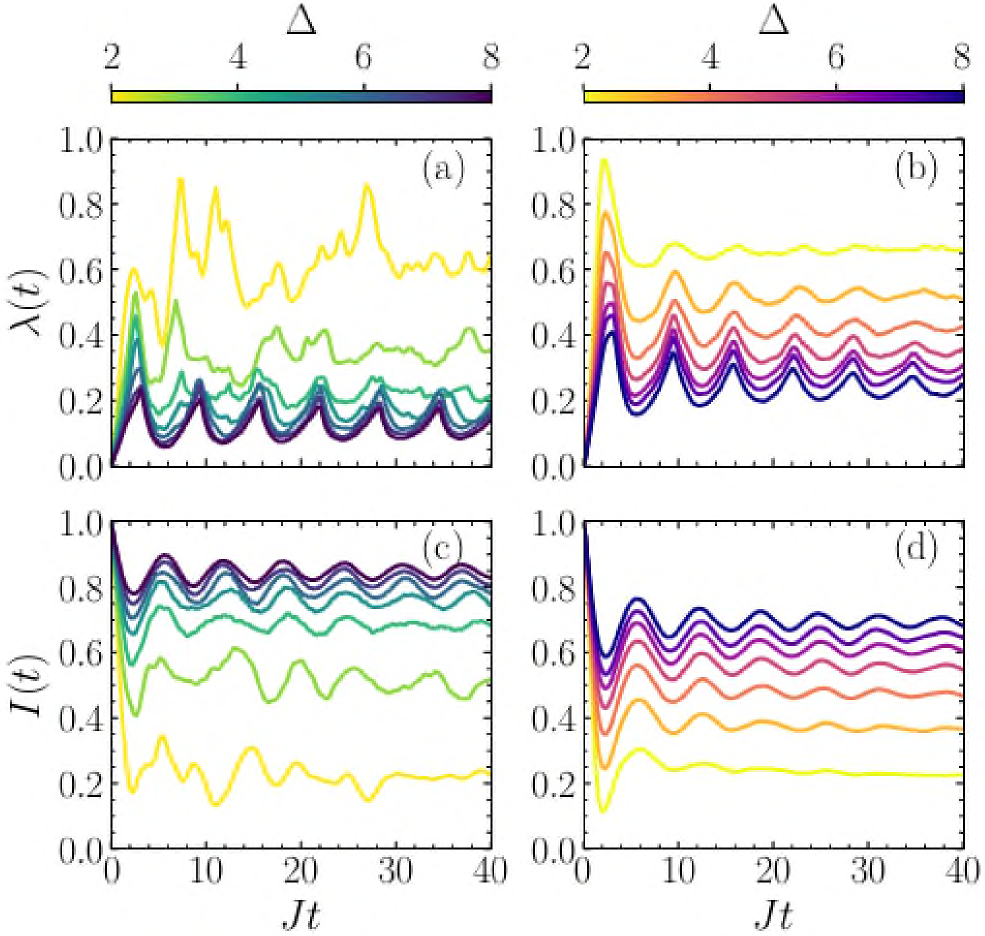
J. Liu, X. Mao, J. Zhong, and R. A. Römer, Ann. Phys. (New York) (2021).

J. Liu, C. Danieli, J. Zhong, R. A. Römer, Phys. Rev. B **106**, 214204 (2022)

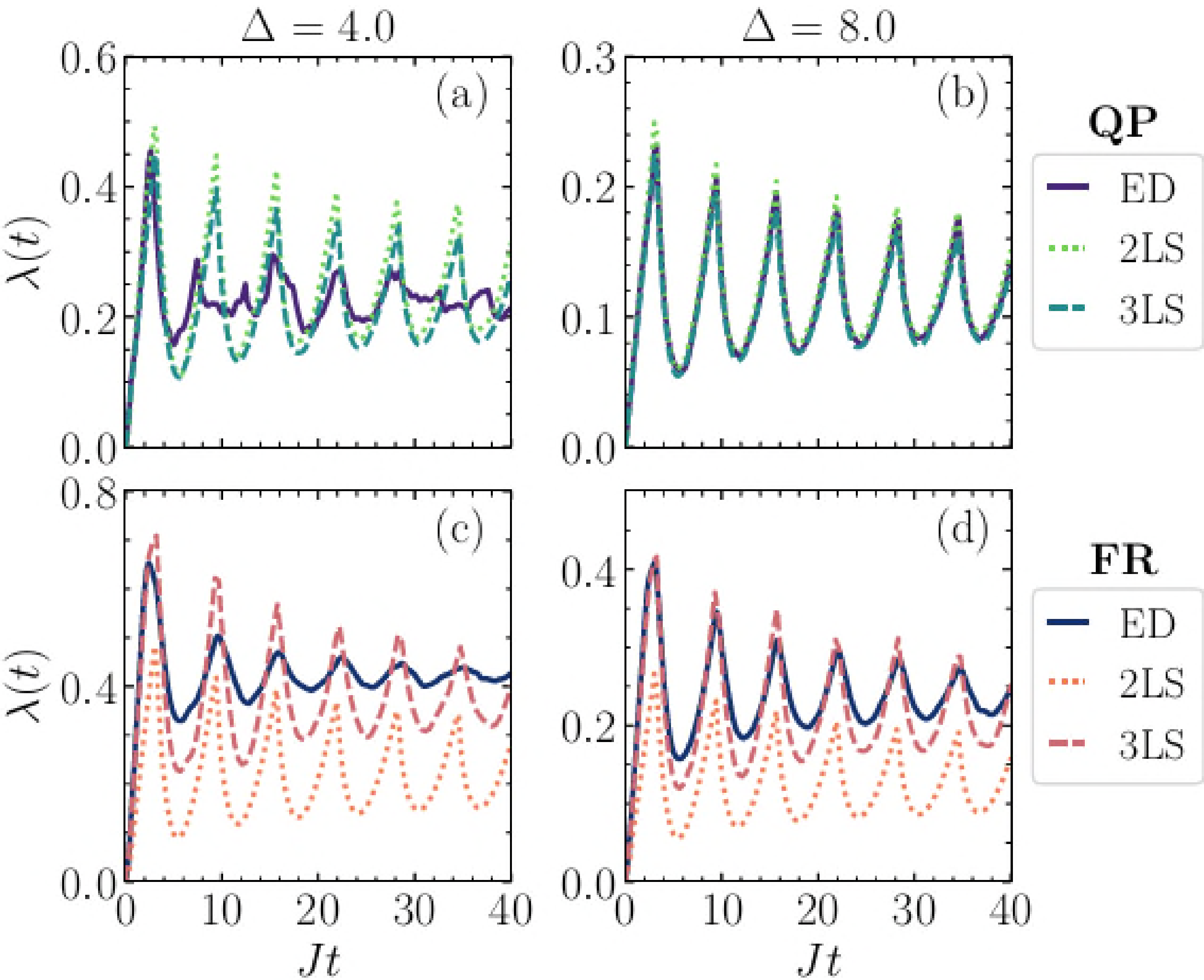
C. Danieli, J. Liu, RAR, submitted to Eur. Phys. J. B, (2023), arXiv:2309.04227

C. Danieli, J. Liu, RAR:

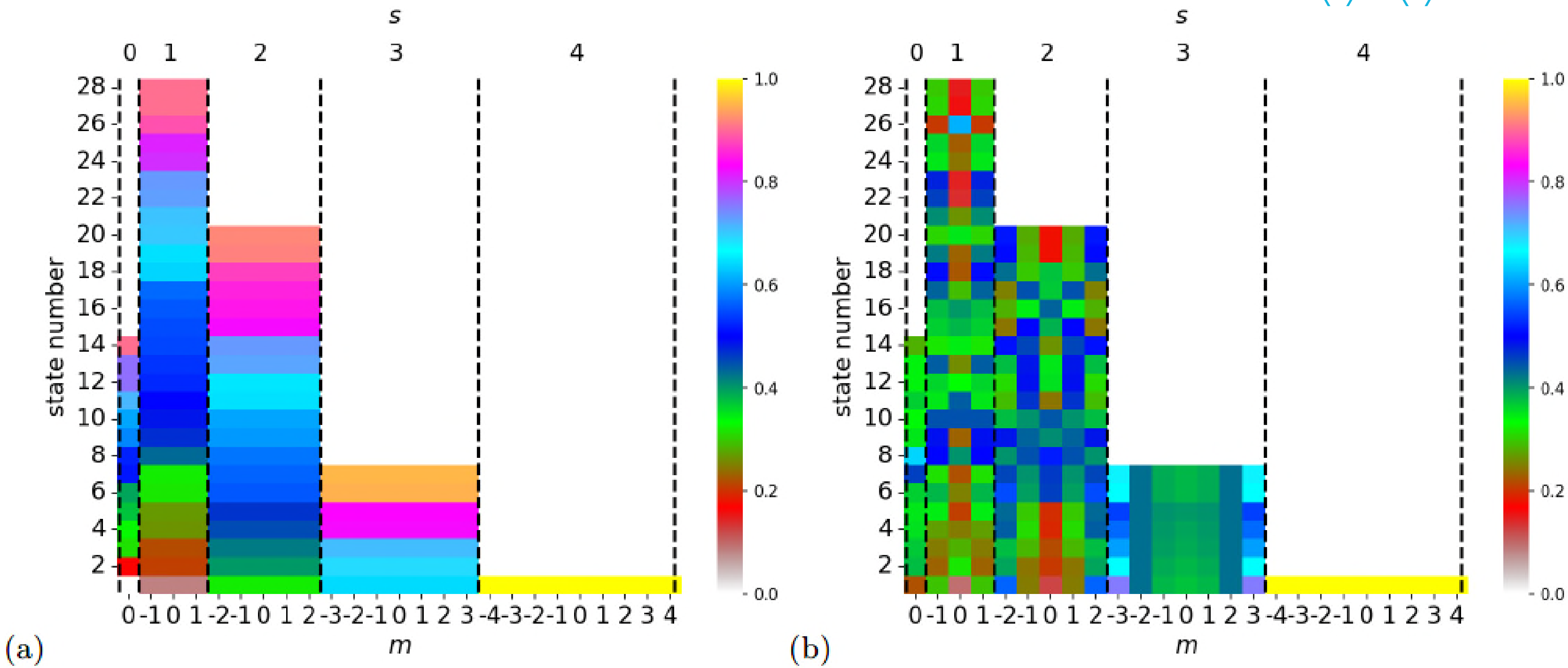




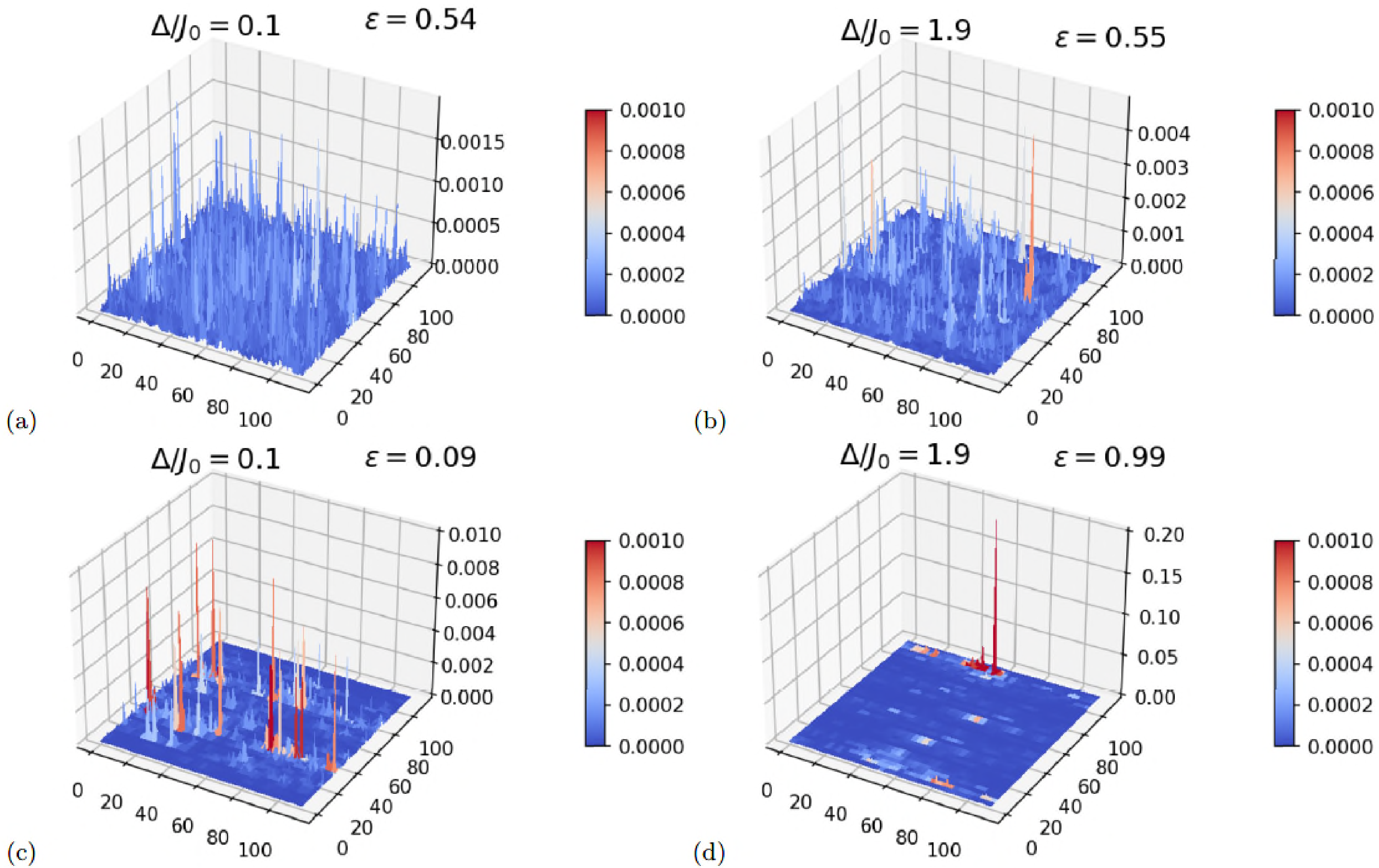
Δ
2
4
6
8
Δ
2
4
6
8
λ(t)
I(t)
(a)
(b)
(c)
(d)
1.0
0.8
0.6
0.4
0.2
0.0
0
10
20
30
40
Jt
Jt



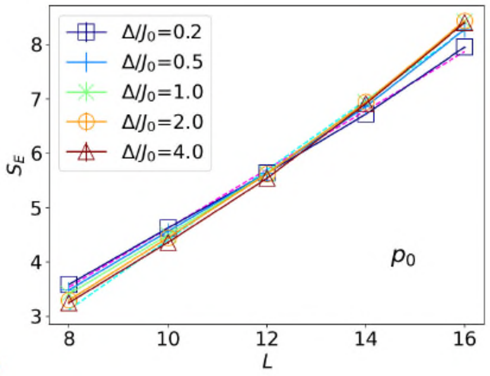
Δ = 4.0
Δ = 8.0
(a)
(b)
(c)
(d)
λ(t)
Jt
QP
ED
2LS
3LS
FR
ED
2LS
3LS



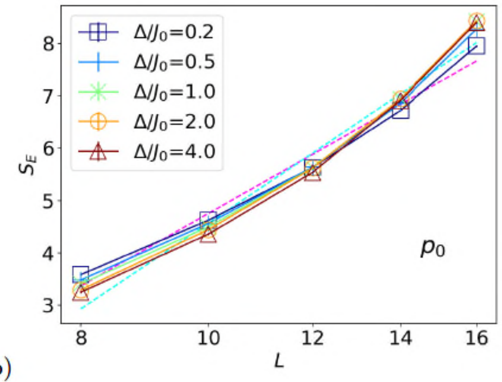
s
0 1 2 3 4
state number
28 26 24 22 20 18 16 14 12 10 8 6 4 2
0 -1 0 1 -2 -1 0 1 2 -3 -2 -1 0 1 2 3 -4 -3 -2 -1 0 1 2 3 4
m
1.0 0.8 0.6 0.4 0.2 0.0
(a)
(b)



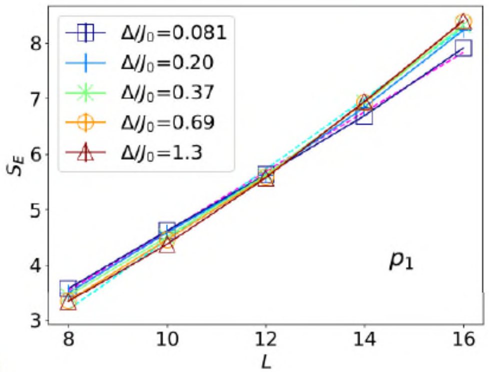
$\Delta/J_0 = 0.1$
$\varepsilon = 0.54$
(a)
$\Delta/J_0 = 1.9$
$\varepsilon = 0.55$
(b)
$\Delta/J_0 = 0.1$
$\varepsilon = 0.09$
(c)
$\Delta/J_0 = 1.9$
$\varepsilon = 0.99$
(d)


Δ/J_0=0.2
Δ/J_0=0.5
Δ/J_0=1.0
Δ/J_0=2.0
Δ/J_0=4.0
p_0
S_E
L

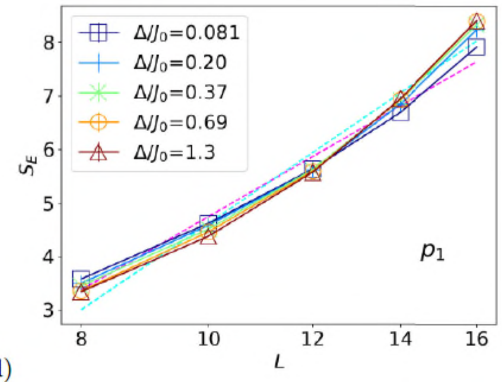
(a)


Δ/J_0=0.2
Δ/J_0=0.5
Δ/J_0=1.0
Δ/J_0=2.0
Δ/J_0=4.0
p_0
S_E
L

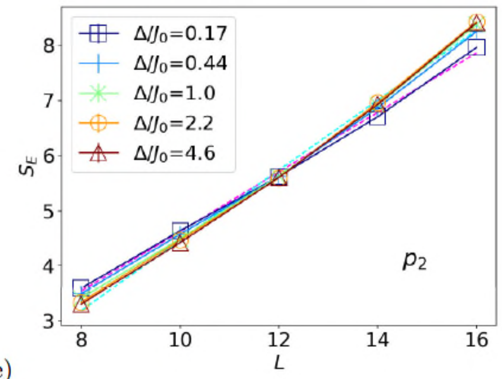
(b)


Δ/J_0=0.081
Δ/J_0=0.20
Δ/J_0=0.37
Δ/J_0=0.69
Δ/J_0=1.3
p_1
S_E
L

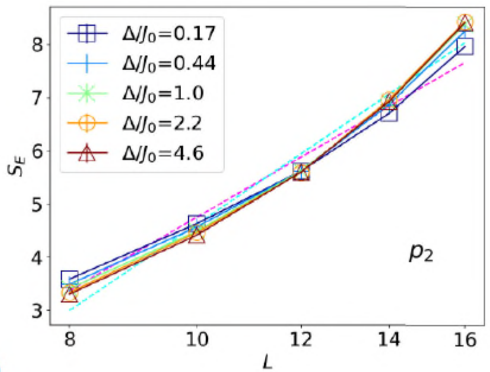
(c)


Δ/J_0=0.081
Δ/J_0=0.20
Δ/J_0=0.37
Δ/J_0=0.69
Δ/J_0=1.3
p_1
S_E
L

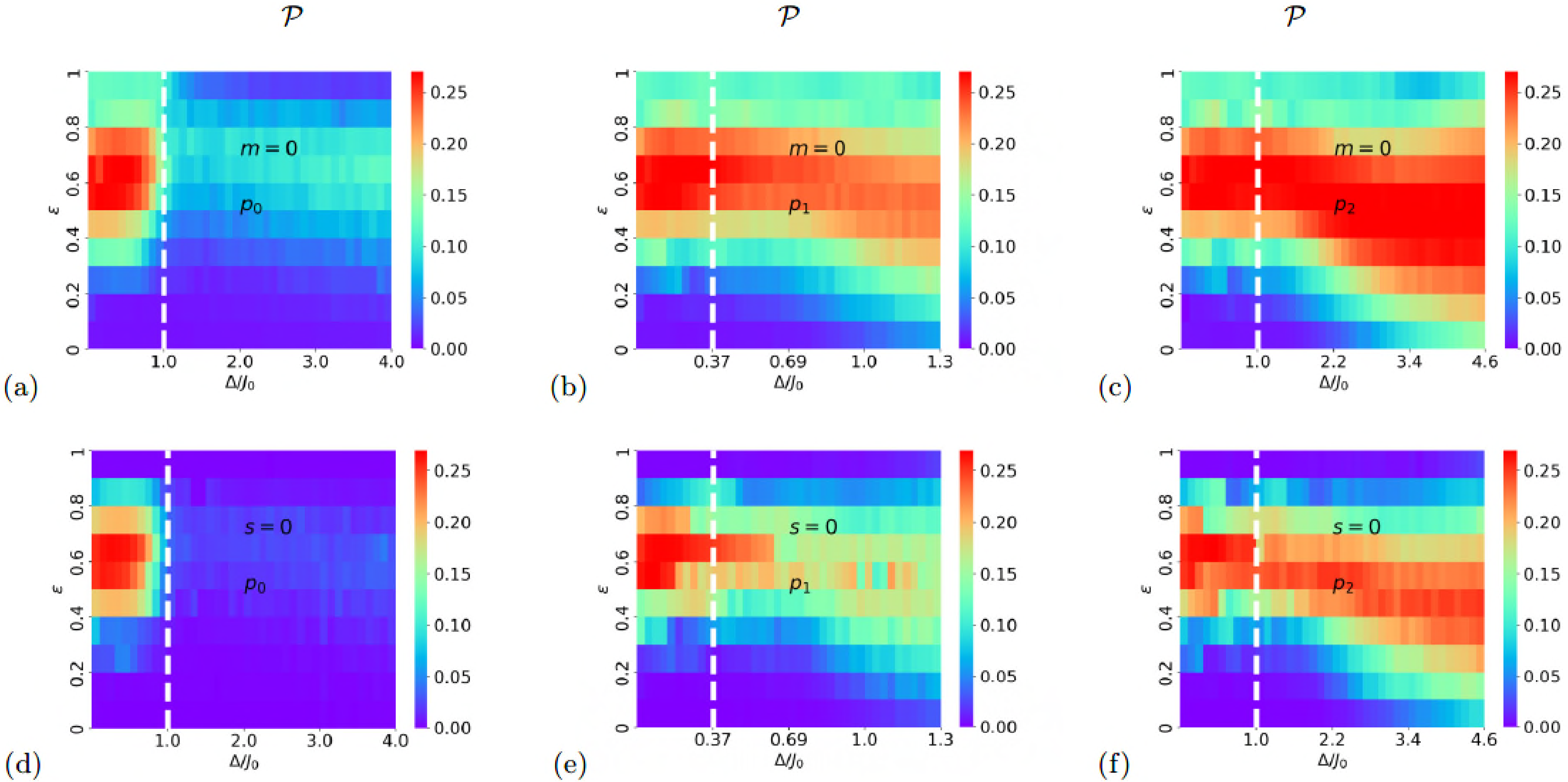
(d)


Δ/J_0=0.17
Δ/J_0=0.44
Δ/J_0=1.0
Δ/J_0=2.2
Δ/J_0=4.6
p_2
S_E
L

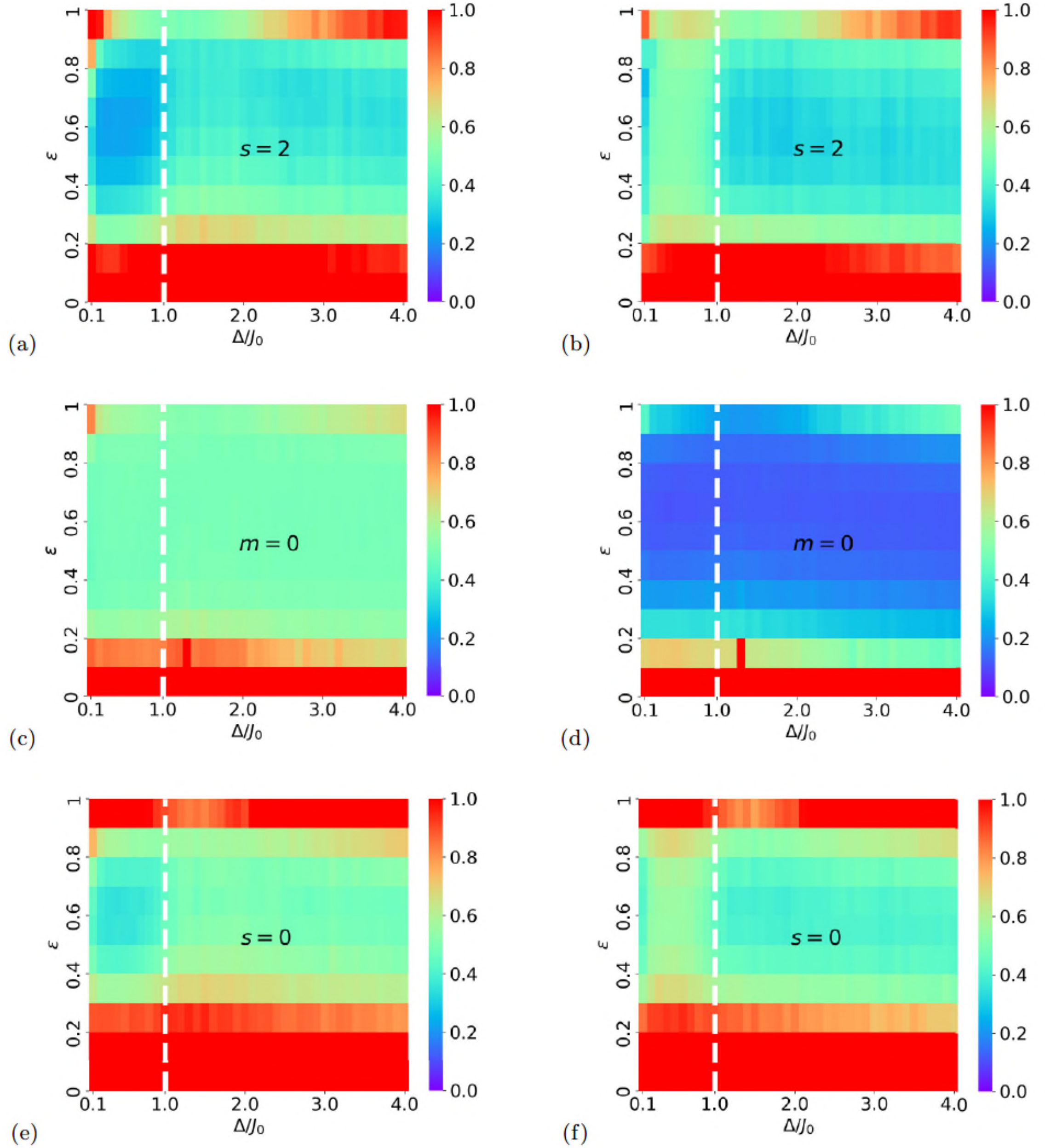
(e)


Δ/J_0=0.17
Δ/J_0=0.44
Δ/J_0=1.0
Δ/J_0=2.2
Δ/J_0=4.6
p_2
S_E
L

(f)



$\mathcal{P}$
$\mathcal{P}$
$\mathcal{P}$
$m = 0$
p_0
p_1
p_2
$s = 0$
ϵ
Δ/J_0
0.00
0.05
0.10
0.15
0.20
0.25
1.0
2.0
3.0
4.0
0.37
0.69
1.3
2.2
3.4
4.6
(a)
(b)
(c)
(d)
(e)
(f)



(a)
(b)
(c)
(d)
(e)
(f)
$s = 2$
$m = 0$
$s = 0$
ε
Δ/J_0
0.1
1.0
2.0
3.0
4.0
0
0.2
0.4
0.6
0.8
1
0.0
0.2
0.4
0.6
0.8
1.0